Analysis and Mathematical Physics (2022) 12: 122
https://doi.org/10.1007/s13324-022-00698-2

Finite orbits of monodromies of rank two Fuchsian systems

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Received: 23 April 2021 / Revised: 21 May 2022 / Accepted: 25 May 2022 /
Published online: 12 September 2022

Abstract
We classified finite orbits of monodromies of the Fuchsian system for five 2×2 matrices. The explicit proof of this result is given. We have proposed a conjecture for a similar classification for 6 or more 2×2 matrices. Cases in which all monodromy matrices have a common eigenvector are excluded from the consideration. To classify the finite monodromies of the Fuchsian system we combined two methods developed in this paper. The first is an induction method: using finite orbits of smaller number of monodromy matrices the method allows the construction of such orbits for bigger numbers of matrices. The second method is a formalism for representing the tuple of monodromy matrices in a way that is invariant under common conjugation way, this transforms the problem into a form that allows one to work with rational numbers only. The classification developed in this paper can be considered as the first step to a classification of algebraic solutions of the Garnier system.

Keywords Fuchian system · Monodromy · Painleve equation · Garnier system

1 Introduction

Let us consider the Fuchsian system for 2×2 matrices:

$$Y(z) \in GL(2, \mathbb{C}) : \frac{d}{dz} Y = \left(\sum_{k=1, a_k \neq \infty}^{n} \frac{A_k}{z - a_k} \right) Y.$$

Here a_k are the branching points, i.e. pairwise distinct numbers on the Riemann sphere, and the following condition for the matrices A_k is implied:

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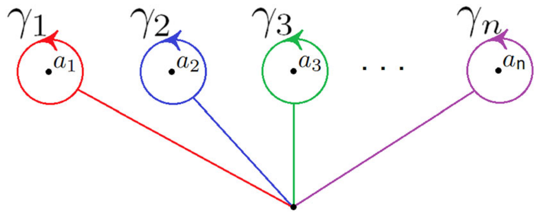
Fig. 1 Monodromy loops



$$\sum_k A_k = 0.$$

Without the loss of generality we can put $Tr\ A_k = 0, \forall k$ and $Y(z) \in SL(2, \mathbb{C})$ for any z, and denote the eigenvalues as follows: $eigen(A_k) = \pm\, \theta_k/2$.

We can perform an isomonodromic deformation for this system, i.e. move the points a_k simultaneously with such evolution of the matrices A_k that the monodromy of Y around the branch points is constant. It gives us the Schlesinger system for 2×2 matrices, or the Garnier system $\mathfrak{g}_{n-3}$ (see [1, 2]), where a_k are independent variables, and the elements of matrices A_k are unknown functions. The number of independent variables is $n-3$, because we can fix three of the points a_k as $0,\ 1,\ \infty$.

Now let us introduce the tuple of monodromy matrices. For this purpose we introduce the collection of loops $\gamma_1 \dots \gamma_n$, as in the Fig. 1. For each loop γ_k there is the monodromy matrix M_k. The product of all monodromy matrices is equal to unity: $M_1\, M_2\, M_3 \dots M_n \;=\; \mathbb{I}$ and the eigenvalues of each monodromy matrix are $eigen(M_k) = \exp(\pm\, i\pi\theta_k)$. Determinant of every monodromy matrix equals 1, due to the fact that the trace of every matrix A is zero.

If the branching points move and interchange with each other, the loops braid and the monodromy matrices are transformed by an action of the braid group (see Figs. 2, 3). We will call this process "braiding of matrices".

The global problem is to classify the algebraic solutions of this system. In this paper we solve a related problem: we classify the finite monodromies of the Fuchsian system which will be a step towards classification of algebraic solutions.

Def: We call the *finite monodromy* a tuple of monodromy matrices that generates only a finite orbit under braid group action up to a common conjugation of matrices. □

In order to define the braid group actions accurately let us introduce some rules and notations. We declare that the case when all the branching points a_k have real positive values is canonical. In this case the loops γ_k are numbered from left to right. If the branching points a_k are not all real positive numbers, then the loops corresponding to them are numbered in order of increasing absolute values $|a_k|$, where the infinity is considered to be the biggest in absolute value. In the case of equal absolute values the corresponding loops are numbered in order of increasing $Arg(a_k)$. Here Arg is the argument of the complex number lying in the interval $(-\pi, \pi]$.

The domain of definition of the Garnier equation is the universal covering of the space of the parameters $a_1 \dots a_n$, which is $(\mathbb{CP}^1)^n$, excluding the cases when any two of the parameters a_k, a_l coincide.

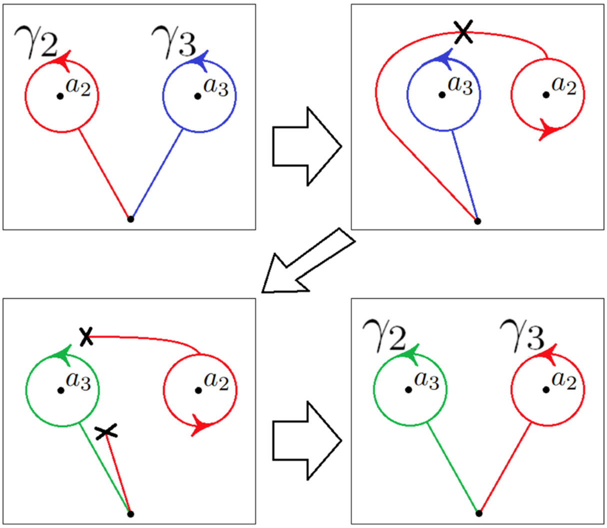


Fig. 2 Braid group action $\mathcal{B}_{23}$ $\{M_2, M_3\} \mapsto \{M_2 M_3 M_2^{-1}, M_2\}$ When the branching points interchange, the corresponding loops braid

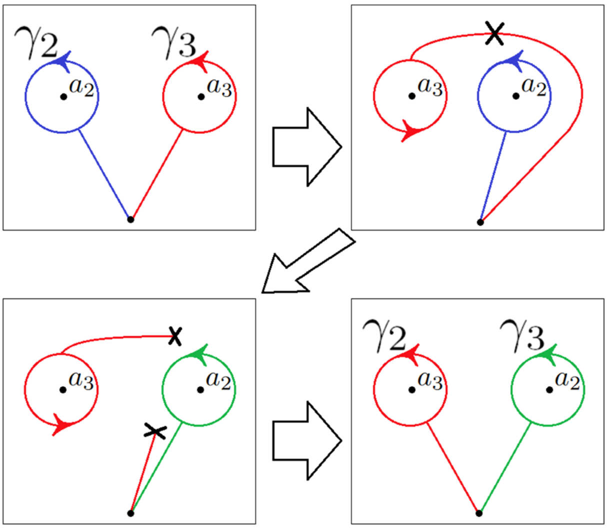


Fig. 3 Braid group action $\mathcal{B}_{32}$ $\{M_2, M_3\} \mapsto \{M_3, M_3^{-1} M_2 M_3\}$

Def: We introduce the term *subbranch*. The universal covering of the space of distinct $a_1, \ldots a_n \in (\mathbb{CP}^1)^n$ can be divided into $n!$ parts, labelled by members of the symmetric group S_n in the following manner: each collection of values $a_1, \ldots a_n$ can be sorted as described above, and the permutation of order of indices correspond to the element s of symmetric group. The exact condition that a point of the universal covering belongs to the part labelled by s is

$$\begin{aligned} & s \in S_n : \quad s(k) < s(l) \leftrightarrow \\ & \quad |a_k| < |a_l| \; OR \; a_l = \infty \; OR \; (|a_k| = |a_l| \; AND \; Arg(a_k) < Arg(a_l)) \,. \end{aligned}$$

Each such part is a disconnected space, and we call the connected components of these parts *subbranches*. □

For every subbranch we can introduce the object $\mathcal{M}^{(n)}$, which is an element of the moduli space of the monodromy.

Def: The object $\mathcal{M}^{(n)}$ is a tuple of n matrices and n integer values:

$$\mathcal{M}^{(n)} : \quad \{M_1, \; M_2 \ldots M_n \, ; \; N_1, \; N_2 \ldots N_n\} \,,$$

where $M_1 \ldots M_n$ are $SL(2, \mathbb{C})$ monodromy matrices, defined up to a common conjugation, the product of all of them is the unity matrix and $N_1 \ldots N_n$ are distinct integer numbers belonging to the interval $[1, n]$. Consequently, the object $\mathcal{M}^{(n)}$ is a member of the following set:

$$\mathcal{M}^{(n)} \in SL(2, \mathbb{C})^{n-1}/SL(2, \mathbb{C}) \; \times \; S_n.$$

Here each matrix M_k is the monodromy matrix corresponding to the loop γ_k, and N_k is an integer defined as follows: if we denote the branching point corresponding to the loop γ_k, as a_m, then $N_k = m$. And if the subbranch which this tuple of monodromy matrices corresponds to, is labeled by the element s of the symmetry group, then γ_k is the loop around the point $a_{s(k)}$, and $N_k = s(k)$.

We call the form of $\mathcal{M}^{(n)}$ with N-values its *long form*, and its form without N-values its *short form*. It will be enough to consider the short form only in the majority of cases treated in the present paper. □

Therefore, the values $N_1 \ldots N_n$ are constant in every subbranch, and the monodromy matrices are constant there up to a common conjugation.

Next, let us define the rule of moving from one subbranch to another.

If two neighboring branching points interchange, then loops corresponding to these points braid and should be redefined to recover the normal form. In this case two corresponding values N interchange, and two matrices M are transformed by braid group action. The braid group action interchanges two matrices and conjugates one of them with another one.

We have two types of such the actions, each interchanges two neighboring branching points, as illustrated in Figs. 2 and 3 respectively:

$$\begin{aligned} & \mathcal{B}_{k,k+1} : \\ & \left\{ \begin{array}{l} M_1, \ldots M_{k-1}, \; M_k, \; M_{k+1}, \; M_{k+2}, \ldots M_n; \\ N_1, \ldots N_{k-1}, \; N_k, \; N_{k+1}, \; N_{k+2}, \ldots N_n \end{array} \right\} \rightarrow \\ & \left\{ \begin{array}{l} M_1, \ldots M_{k-1}, \; M_k M_{k+1} M_k^{-1}, \; M_k, \; M_{k+2}, \ldots M_n; \\ N_1, \ldots N_{k-1}, \; N_{k+1}, \; N_k, \; N_{k+2}, \ldots N_n \end{array} \right\}, \end{aligned}$$

$$\mathcal{B}_{k+1,k}:$$
$$\left\{\begin{matrix} M_1, \dots M_{k-1}, M_k, M_{k+1}, M_{k+2}, \dots M_n; \\ N_1, \dots N_{k-1}, N_k, N_{k+1}, N_{k+2}, \dots N_n \end{matrix}\right\} \to$$
$$\left\{\begin{matrix} M_1, \dots M_{k-1}, M_{k+1}, M_{k+1}^{-1}M_kM_{k+1}, M_{k+2}, \dots M_n; \\ N_1, \dots N_{k-1}, N_{k+1}, N_k, N_{k+2}, \dots N_n \end{matrix}\right\},$$

or briefly

$$\mathcal{B}_{k,k+1}: \quad M_k \to M_kM_{k+1}M_k^{-1}, \quad M_{k+1} \to M_k, \quad N_k \leftrightarrow N_{k+1}, \tag{1}$$
$$\mathcal{B}_{k+1,k}: \quad M_k \to M_{k+1}, \quad M_{k+1} \to M_{k+1}^{-1}M_kM_{k+1}, \quad N_k \leftrightarrow N_{k+1}. \tag{2}$$

The two indices of $\mathcal{B}$ are the numbers of braided loops and must differ by ± 1. Informally we will call the braid group actions *braiding*.

It should be noticed that all features of the tuple of monodromy matrices are symmetric under cyclic permutation of matrices and N-values, hence index k of M_k matrix may be treated as an integer modulo n. Thus, in total we have $2\,n$ braid group actions.

Armed with these definitions, let us proceed to formulating the problem of finite orbits of the tuples of monodromy matrices under braid group actions.

If the solution of the Garnier system is algebraic, then it has a finite branching. Thus in order to classify the algebraic solutions we have to classify the finite orbits of the braid group acting on monodromies.

The goal of this paper is a classification of all finite orbits of the braid group action on monodromies.

Def: We call the $\mathcal{M}^{(n)}$ *triangular* if all matrices have a common eigenvector, therefore can be made simultaneously to be lower-triangular. □

Def: The *orbit* is the set of all $\mathcal{M}^{(n)}$'s, obtained from one of them, by the action of the braid group. □

We denote the $\mathcal{M}^{(n)}$ which belongs to a finite orbit with a subscript F: $\mathcal{M}_F{}^{(n)}$. Note that $\mathcal{M}^{(n)}$ and $\mathcal{M}_F{}^{(n)}$ refer to fixed tuples of matrices, not to sets of all such tuples. □

Def: We will call the operation of replacing of two neighboring matrices by their product (using the short form) the *reduction* from a $\mathcal{M}^{(n)}$ to $\mathcal{M}^{(n-1)}$. □

Def: The inverse operation is the following: replacing any matrix by two matrices where one of them is an arbitrary $SL(2, \mathbb{C})$ matrix and another one is such that their product equals the original matrix. We will call this to be an *induction* from a $\mathcal{M}^{(n)}$ to $\mathcal{M}^{(n+1)}$. A special type of induction is an *addition of the unit matrix*. □

The *length* of the orbit is defined as the number of branches in the transformation of monodromy. In terms of $\mathcal{M}^{(n)}$ the length is the number of such members of the orbit that their N_k's have trivial permutation: $N_k = k, \ \forall k$. Therefore, the total number of members of the orbit is $length \times n!$. This definition of the length is the only reason to define $\mathcal{M}^{(n)}$ to be not simply a tuple of matrices, but a tuple of matrices associated with a tuple of numbers. The number of members of the orbit with different tuples of matrices (not taking into account the permutations) is the number which is a multiple of the length and a divisor of $length \times n!$. Nevertheless, we do not define the length by this number, because such the definition would be mathematically unnatural.

Our problem is classification of all finite orbits. In this paper the problem has been solved for $\mathcal{M}^{(5)}$, excluding the *triangular* case.

It is important to note that there are some symmetries for $\mathcal{M}^{(n)}$ which are not equivalences: the cyclical permutation of all monodromy matrices (see 3); multiplying any two matrices by -1 (see 4); taking inverses of all matrices, simultaneously with reversing their order (see 5); complex conjugation of all elements of all matrices (this symmetry will not be used in the paper):

$$M_1,\ M_2 \dots M_n \to M_2,\ M_3 \dots M_n,\ M_1; \tag{3}$$

$$M_1,\ \dots M_p,\ \dots M_q,\ \dots M_n \to M_1,\ \dots -M_p,\ \dots -M_q,\ \dots M_n; \tag{4}$$

$$M_1,\ M_2 \dots M_n \to M_n^{-1},\ M_{n-1}^{-1} \dots M_1^{-1}. \tag{5}$$

Note that operation (3) can be represented as a series of braidings $\mathcal{B}_{1,2}$, $\mathcal{B}_{2,3}$,...$\mathcal{B}_{n-1,n}$. That is why every orbit is closed under this operation.

For $n \leq 3$ the problem formulated above is trivial: any $\mathcal{M}^{(1)}$, $\mathcal{M}^{(2)}$ and $\mathcal{M}^{(3)}$ belong to the orbit of the length 1. Also, there exists a trivial method of transforming of ${\mathcal{M}_F}^{(n)}$ into $\mathcal{M}^{(n+1)}$ (which also belongs to a finite orbit): this is achieved by addition of the unit matrix. Note that for any n there exists a trivial case when all monodromy matrices commute.

The problem posed above is solved for the case of $n = 4$ in [3]. That solution is used in the present paper as a base for the construction finite orbits of $\mathcal{M}^{(5)}$'s.

2 Signature formalism

Now we choose the most convenient formalism for classification of ${\mathcal{M}_F}^{(n)}$'s—elements of the moduli space of the monodromy, which belong to finite orbits.

First, we note that it is better to notate ${\mathcal{M}_F}^{(n)}$'s not by elements of the matrices, but by traces of different products of these matrices. This is because the common conjugation, which is by definition a trivial transformation at the moduli space of the monodromy, acts trivially on these traces, but not on the elements of matrices.

Further, since we are looking for $\mathcal{M}^{(n)}$'s belonging to finite orbits, the products of their matrices are likely to be roots of unity, so their traces are likely to have a form

$$2\cos(\pi\mathbb{Q}),$$

where $\mathbb{Q}$ means the set of rational numbers. Since for rigorous classification it is better to operate with rational numbers, we propose a described below formalism.

Def: *Signature*. For each $\mathcal{M}^{(n)}$ a special collection of values can be calculated. It is called the *signature*.

The *signature* consists of sub-collections of the following values:

(1) the θ value for any matrix:

$$\forall x \in [1, n]: \ \theta_x = \frac{1}{\pi} \arccos \frac{1}{2} Tr\,(M_x)\,, \tag{6}$$

where the eigenvalues of M_x are $\exp(i\pi\theta_x)$ and $\exp(-i\pi\theta_x)$.

(2) the σ value for any subsequence of two or more neighboring matrices in $\mathcal{M}^{(n)}$

$$\forall x \in [1, n]\ \forall y \in [x+1, x+n-2]: \ \sigma_{x,\,x+1\,\ldots y} = \frac{1}{\pi} \arccos \frac{1}{2} Tr \left(\prod_{z=x}^{y} M_{z\,mod\,n} \right), \tag{7}$$

where the product $\prod_{z=x}^{y} M_{z\,mod\,n}$ has eigenvalues $\exp(i\pi\sigma_{x\ldots y})$ and $\exp(-i\pi\sigma_{x\ldots y})$.

(3) The σ value for any two not intersecting subsequences of neighboring matrices is the following one:

$$\begin{gathered} \forall x \in [1, n]\ \forall y \in [x, x+n-4]\ \forall p \in [y+2, x+n-2]\ \forall q \in [p, x+n-2]: \\ \sigma_{x,\,x+1,\,\ldots y,\,p,\,p+1,\,\ldots q} = \frac{1}{\pi} \arccos \frac{1}{2} Tr \left(\prod_{z=x}^{y} M_{z\,mod\,n} \prod_{z=p}^{q} M_{z\,mod\,n} \right). \end{gathered} \tag{8}$$

Anyway, every θ or σ depends on the trace of a product of matrices, and the tuple of indices of θ or σ means the indices of these matrices.
Due to the definition, all θ's and σ's are determined modulo 2 and up to a sign. The order of indices in σ is important, but the cyclical permutation of the indices is treated as an equivalence:

$$\sigma_{a,b,\ldots c} \equiv \sigma_{b,\ldots c,a}; \quad \sigma_{a,c,b} \neq \sigma_{a,b,c}.$$

The following notations are also treated as equivalent:

$$\theta_x \equiv \sigma_{x+1,\ldots x+n-1}, \quad \sigma_{x,x+1,\ldots y} \equiv \sigma_{y+1,y+2,\ldots x+n-1}. \tag{9}$$

Indeed, $M_x \cdot M_{x+1} \cdot \ldots \cdot M_y = \left(M_{y+1} \cdot \ldots \cdot M_{x+n-1}\right)^{-1}$, so the corresponding traces are equal. □

We call the signature of $\mathcal{M}^{(n)}$ to be n-signature. The reason of using such formalism is that θ and σ values for a $\mathcal{M}_F$ are almost always rational numbers.

We call the signature *inconsistent*, if no tuple of matrices which corresponds to it exists.

Each allowed tuple of indices in the signature we call the *cell*. If there is one index (corresponding to a trace of one monodromy matrix)—the cell is called θ, if there are more indices than one—the cell is called σ. We say that two tuples of indices $(x,\ x+1,\ \ldots y)$, $(y+1,\ y+2,\ \ldots x+n-1)$ or x, $(x+1,\ \ldots x+n-1)$ according to (9) define the same cell.

For example in the case of $\mathcal{M}^{(4)}$, denoted as

$$M_1,\ M_2,\ M_3,\ M_4,\ \ M_1M_2M_3M_4 = \mathbb{I}$$

the signature consists of eight numbers:

$$\theta_1,\ \theta_2,\ \theta_3,\ \theta_4,\ \sigma_{12},\ \sigma_{23},\ \sigma_{13},\ \sigma_{24}.$$

In the case of $n = 5$ the signature consists of 20 cells, in the case of $n = 6$ it consists of 39 cells, and in general for any n it consists of $n(n-1)(n^2-5n+12)/12$ cells: n of them are θ's and $n(n-3)(n^2-3n+8)/12$ of them are σ's.

Def: We call by the *particular signature* a signature in which some cells are undefined: no values for these cells are defined. □

Def: We call by the *incomplete signature* a special case of the particular signature when for some indices a, b such that $b - a = \pm 1$ all the σ's which contain index a and do not contain index b, are undefined, but the rest of cells are defined. In case if there are two notations for one cell, (see 9) and at least one of them does not contain a or contains b, then this cell must be defined. □

In the incomplete signature all the θ's are defined, including the θ_a: although in the θ_a the index a is present and the index b is absent, but the θ_a can be notated as $\sigma_{a+1,\ldots a-1}$.

In the $n = 5$ case the incomplete signature contains 16 cells.

The example of an incomplete signature for $n = 4$ with $a = 1$ and $b = 2$ is this: we take all cells despite those which contain index 1 and do not contain index 2. The $\sigma_{12} = \sigma_{34}$ is defined; the σ_{14} seems to be undefined, but it is equal to σ_{23}, which is defined. The σ_{13} is undefined, and the σ_{24} is defined. In total, this incomplete signature consists of seven values, and as it is known, it is enough to reconstruct all matrices up to common conjugation, except for the *triangular* case.

We call by *merging* of two particular signatures a procedure of making of them one signature or particular signature by filling in the cells. It is impossible to merge two particular signatures if they have at least one cell which is defined in both particular signatures, but have different values. Value of the cell which is defined in both merging particular signatures and coincide, or defined only in one—is retained; the cell which is undefined in both signatures—remains undefined.

We say that two signatures, incomplete signatures or particular signatures *coincide* if in all cells, which are defined in both of them, the values coincide.

Lemma 1 *The signature (and also the incomplete signature) is sufficient for unique reconstruction of $\mathcal{M}^{(n)}$ (up to simultaneous conjugation), except for the triangular case (where all matrices will have a common eigenvector, therefore can be made simultaneously to be lower-triangular, because in this case the* $[2, 1]$ *elements of matrices can have arbitrary values and do not affect the signature).*

Proof To generalize the proof for signature and incomplete signature, we re-formulate the problem: there is a linear tuple of matrices, we know the traces of every matrix, of every product of subsequence of neighboring matrices, and of the product of each two non-intersecting subsequences. For an incomplete signature, in which there are undefined cells containing index a and not containing b, we consider a linear tuple of all matrices except for M_a; then, after the reconstruction of all other matrices, we will be able to reconstruct M_a using the fact that the product of all matrices must make the unit matrix.

To prove the lemma, let us consider three cases:

Case 1: The general case. In this case, there exists at least one pair of neighboring matrices which have no common eigenvector. Let us call them M_p and M_q. The non-existence of a common eigenvector can be checked using the condition

$$Tr(M_p)^2 + Tr(M_q)^2 + Tr(M_p \cdot M_q)^2 \neq Tr(M_p)Tr(M_q)Tr(M_p \cdot M_q) + 4, \tag{10}$$

taking into account that both matrices have determinant 1.

Case 1a: The most general case is the most simple one: one of the above two matrices, let's M_p, has the trace not equal to $\pm\, 2$.

The M_p can be diagonalized. We make simultaneous conjugation of all the matrices such that M_p turns out to be diagonal with different diagonal elements.

Now we know all diagonal elements of all matrices: for any $r \neq p$ we have $M_r[1,1] + M_r[2,2] = Tr(M_r)$ and $M_r[1,1]M_p[1,1] + M_r[2,2]M_p[2,2] = Tr(M_r\, M_p)$. This is the system of two equations for $M_r[1,1]$ and $M_r[2,2]$. The system has a unique solution because the matrix elements $M_p[1,1]$ and $M_p[2,2]$ are the eigenvalues of M_p and are distinct by our assumption.

Then, due to the condition that M_p and M_q have no common eigenvectors, the non-diagonal matrix elements $M_q[1,2]$ and $M_q[2,1]$ are both nonzero, and we can perform such the conjugation that M_p remains diagonal and $M_q[2,1]$ turns 1.

Now we can reconstruct the non-diagonal elements of any other monodromy matrix, let us call it M, using the non-degenerate system of two linear equations:

$$\begin{aligned} &M[1,2]M_q[2,1] + M[2,1]M_q[1,2] \\ &\quad = Tr(M_q \cdot M) - M_q[1,1]M[1,1] - M_q[2,2]M[2,2],\ M[1,2]M_p[2,2]M_q[2,1] \\ &\qquad + M[2,1]M_p[1,1]M_q[1,2] \\ &\quad = Tr(M_p \cdot M_q \cdot M) - M_p[1,1]M_q[1,1]M[1,1] - M_p[2,2]M_q[2,2]M[2,2]. \end{aligned}$$

Case 1b: Similar calculations can be performed in the case when M_p and M_q both have traces equal to $\pm\, 2$:

Let us consider the case $Tr M_p = 2$. This matrix can't be unity, because it has no common eigenvectors with M_q. That is why it is possible, using the common conjugation, to put M_p to lower triangular form, with all non-zero elements equal to 1. Since matrices have no common eigenvector then $M_q[1,2] \neq 0$. Moreover, we can select such a conjugation that $M_q[1,1] = 0$, and then $M_q[2,2] = \pm\, 2$ and $M_q[2,1] = -1/M_q[1,2]$. Now we can reconstruct all elements of any other matrix M using the system of equations:

$$M[1,2] = Tr(M \cdot M_p) - Tr(M),$$
$$M[2,2]M_q[1,2] = Tr(M \cdot M_p \cdot M_q) - Tr(M \cdot M_q),$$

and take $M[1,1]$ from TrM, and the $M[2,1]$—from $Tr(M \cdot M_q)$.

Case 2: There exist two non-intersecting subsequences of neighboring matrices $M_p, ..., M_q$ and $M_s, ..., M_t$ ($p \leq q < s \leq t$), such that products $M_p \cdot ... \cdot M_q$ and $M_s \cdot ... \cdot M_t$ have no common eigenvector.

Let us construct matrices $M_p \cdot ... \cdot M_q$ and $M_s \cdot ... \cdot M_t$ similarly to the Case 1. For any $r < p$ we reconstruct the elements of the matrix which is the product $M_r \cdot ... \cdot M_{p-1}$ similarly to Case 1, knowing the values of the following traces: $Tr(M_r \cdot ... \cdot M_{p-1})$, $Tr(M_r \cdot ... \cdot M_{p-1} \cdot M_p \cdot ... \cdot M_q)$, $Tr(M_r \cdot ... \cdot M_{p-1} \cdot M_s \cdot ... \cdot M_t)$, $Tr(M_r \cdot ... \cdot M_{p-1} \cdot M_p \cdot ... \cdot M_q \cdot M_s \cdot ... \cdot M_t)$, exact form of the matrices $M_p \cdot ... \cdot M_q$ and $M_s \cdot ... \cdot M_t$ and the condition that they have no common eigenvector.

Hence we can reconstruct all matrices M_r for $r < p$.

In similar manner we can reconstruct matrices M_r for $q < r < s$ and $r > t$.

Finally, for every $r \in [p, q)$ we can reconstruct the matrix $M_p \cdot ... \cdot M_r$ using the known values of four traces: $Tr(M_p \cdot ... \cdot M_r)$, $Tr(M_p \cdot ... \cdot M_r \cdot M_s \cdot ... \cdot M_t)$,

$$Tr(M_p \cdot ... \cdot M_r \cdot M_p \cdot ... \cdot M_q) = Tr(M_p \cdot ... \cdot M_r)$$
$$Tr(M_p \cdot ... \cdot M_q) - Tr(M_{r+1} \cdot ... \cdot M_q)$$

and in a similar way we calculate the value of $Tr(M_p \cdot ... \cdot M_r \cdot M_p \cdot ... \cdot M_q \cdot M_s \cdot ... \cdot M_t)$. This is the linear system of four equations for four variables (four elements of the matrix $(M_p \cdot ... \cdot M_r)$) which has a unique solution due to the fact that two matrices $(M_p \cdot ... \cdot M_q)$ and $(M_s \cdot ... \cdot M_t)$ have no common eigenvector.

Therefore we know all matrices M_r for $r \in [p, q]$ and, after the same procedure, for $r \in [s, t]$.

Case 3: Every two matrices and every two products of non-intersecting subsequences of neighboring matrices have a common eigenvector.

This case can be *triangular* only: all matrices have a common eigenvector.

Indeed, assume that this assumption is wrong.

First, in this case a monodromy matrix with only one eigenvector cannot exist because this eigenvector would be a common for all the matrices.

Therefore, if there exist matrices with traces equal to ± 2, then they are proportional to unit matrix (are scalar matrices), and we can exclude them, retaining our knowledge about traces, and equivalently considering the case without these matrices. If all matrices are scalar ones, then this contradicts the assumption of non-existence of common eigenvector of all matrices, so we will assume that there exists at least one non scalar matrix.

Then there remain only matrices with traces different from ± 2.

The matrix M_1 can be made diagonal. Its eigenvectors are v_1 and v_2 (see 11). For all other matrices one of these vectors is an eigenvector, but there exists at least one matrix, let us call it M_p, for which v_1 isn't eigenvector (but v_2 of course is), and at least one other matrix, let us call it M_q, for which v_2 isn't eigenvector (but v_1 is). Therefore, after excluding scalar matrices, the tuple contains at least three matrices M_1, M_p and M_q. The matrices M_p and M_q must have a common eigenvector, let us call it v_3, and, using the common conjugation not affecting v_1 and v_2, we can transform this vector to the form (11):

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{11}$$

Therefore, any other matrix must have a common eigenvector with M_1, M_p and M_q, and that is why the eigenvectors of every matrix must be two of the three vectors v_1, v_2 and v_3.

It means that every matrix must belong to the one of three types (12), and for each type there exists at least one matrix:

$$\begin{gathered} M = \begin{pmatrix} e^{i\pi\theta} & 0 \\ 0 & e^{-i\pi\theta} \end{pmatrix}, \quad M = \begin{pmatrix} e^{i\pi\theta} & 0 \\ e^{i\pi\theta} - e^{-i\pi\theta} & e^{-i\pi\theta} \end{pmatrix}, \\ M = \begin{pmatrix} e^{i\pi\theta} & e^{-i\pi\theta} - e^{i\pi\theta} \\ 0 & e^{-i\pi\theta} \end{pmatrix}. \end{gathered} \tag{12}$$

Then there must exist at least one pair of neighboring matrices belonging to different types. Let it be M_r with the eigenvectors v_1 and v_2, and M_{r+1} with the eigenvectors v_1 and v_3. But their product does not belong to any of three above types (to see this we take into account that traces of all matrices differ from $\pm$ 2). The matrix $(M_r \cdot M_{r+1})$ has the eigenvector v_1, but not v_2 or v_3. By the way, there exists at least one matrix of such type that its eigenvectors are v_2 and v_3. This is M_p. It has no common eigenvectors with $(M_r \cdot M_{r+1})$.

Therefore the above assumption is wrong and it means that the considered tuple is *triangular*: all matrices in the considered tuple have a common eigenvector.

Lemma 1 has been proven. □

Notice: There exist inconsistent signatures for which a tuple of monodromy matrices does not exist.

For every signature, or incomplete signature, we have three possibilities:

1. tuple of matrices does not exist (the signature is inconsistent),
2. there exists only one tuple of matrices and it has no common eigenvector for all matrices (non-triangular case),
3. there exist many tuples of matrices, such that all matrices have common eigenvector and can be simultaneously made lower-triangular (triangular case).

At a first glance, the signature contains excessive information about the collection of matrices $\mathcal{M}^{(n)}$. Indeed, due to its definition the collection of matrices $\mathcal{M}^{(n)}$ has only $3n - 6$ degrees of freedom. But the corresponding signature contains $n(n-1)(n^2 - 5n + 12)/12$ cells.

But if we use smaller tuple of σ's than defined in (7,8)—several discrete options for reconstruction of the tuple of matrices remain. For example for the tuple $\mathcal{M}^{(4)}$ we have $4 \times 3 - 6 = 6$ degrees of freedom. But if we use only six cells of the signature—four

θ's and two σ's—two options remain. Anyway any smaller tuple of σ's will be not enough for the Lemma 1.

This formalism is similar to formalism of p values developed in [4], and every σ from the present formalism is

$$\sigma = \frac{1}{\pi} \arccos(\frac{p}{2}).$$

.

3 The list of signatures of ${\mathcal{M}_F}^{(4)}$'s

Here we present the list of 4-signatures which correspond to ${\mathcal{M}_F}^{(4)}$'s.

The present list is obtained by our computer program, and was compared with list in the paper [3] (Theorem 1 and Table 4) to make sure that it is obtained correctly.

To shorten the list we present only one member of each orbit.

The present list differs from the list of [3] in the following four aspects: in majority of orbits another element of the orbit is presented; here we present θ's, while two or three different tuples of θ's can correspond the same tuple of $\omega_x,\ \omega_y,\ \omega_z,\ \omega_4$. That is why the list turned out to be almost three times longer; notations $\sigma_{23},\ \sigma_{13},\ \sigma_{12}$ are used instead of $r_x,\ r_y,\ r_z$; σ_{24} is also presented.

Therefore, in this list, see Table 1 a signature with three parameters is presented, which corresponds to any triangular tuple (it can correspond to different orbits of $\mathcal{M}^{(4)}$'s); orbit 2 is obtained from the orbit of $\mathcal{M}^{(3)}$ with three arbitrary parameters by addition of one unit matrix; then, there are the orbits $3 - 7$ with two or one arbitrary parameter; there are also orbits $8,\ 9$ with two rational parameters, each orbit have the length depending on the common denominator of its parameters (we write an estimation of the length instead of the exact formula): $4\,denominator^2/\pi^2 < length < denominator^2/2 + 1$; and there are the orbits $10 - 131$ with the following explicit values. The list is presented in the Table 1.

4 Construction of $\mathcal{M}^{(n)}$'s of higher order

Lemma 2 *For each* ${\mathcal{M}_F}^{(n)}$ *which will be notated as*

$$M_1,\ M_2,\ M_3 \ldots M_n$$

its reduction

$$(M_1 \cdot M_2),\ M_3 \ldots M_n$$

is a ${\mathcal{M}_F}^{(n-1)}$*—it also belongs to a finite orbit.*

Proof Here we use the long form of the $\mathcal{M}^{(n)}$. The starting point of the orbit we denote

$$M_1,\ M_2,\ \ldots M_n, \quad N_1 = 1,\ N_2 = 2,\ \ldots N_n = n.$$

Table 1 List of 4-signatures which generate finite orbits

♯	Length	θ_1	θ_2	θ_3	θ_4	σ_{12}	σ_{23}	σ_{13}	σ_{24}
1		x	y	z	$x+y+z$	$x+y$	$y+z$	$x+z$	$x+z$
2	1	x	y	z	0	z	x	y	y
3	2	x	1/2	y	1/2	1/2	1/2	$x+y$	$x-y$
4	2	x	y	x	$y+1$	1/2	1/2	$2x$	$2y+1$
5	3	$2x$	x	x	2/3	1/2	1/3	1/2	$3x$
6	4	x	x	x	$3x+1$	1/3	1/3	1/3	$4x+1$
7	4	x	x	x	1/2	1/3	1/3	1/3	$2x$
8		1/2	1/2	1/2	1/2	z	y	$z+y+1$	$z-y+1$
9		0	0	0	1	z	y	$z+y+1$	$z-y+1$
10	5	7/15	2/5	13/15	2/5	1/2	1/2	1/2	1/2
11	5	1/5	2/5	2/5	1/5	1/3	0	1/2	1/2
12	5	2/5	1/3	4/5	1/3	1/2	1/2	1/2	1/2
13	5	1	1	4/5	2/5	0	1/3	1/2	1/2
14	5	14/15	4/5	11/15	1/5	1/2	1/2	1/2	1/2
15	6	2/3	1/2	2/3	1/2	2/3	2/3	1/2	1/2
16	6	3/4	2/3	1/2	1/2	1/2	1/3	1/2	1/2
17	6	17/24	13/24	13/24	7/24	2/3	1/2	1/2	1/2
18	6	1	5/12	1/12	0	1/4	1	1/2	1/2
19	6	1/4	1/3	1/3	1/4	1/4	0	1/2	1/2
20	6	19/24	19/24	23/24	1/24	1/2	1/3	1/2	1/2
21	6	1/6	1	1/6	0	1/3	2/3	1/2	1/2
22	6	2/5	2/5	2/3	1/5	3/5	3/5	1/2	1/2
23	6	4/5	2/3	2/5	1/5	1/5	4/5	1/2	1/2
24	6	5/6	19/30	11/30	7/30	1/5	4/5	1/2	1/2
25	6	17/30	5/6	19/30	13/30	3/5	2/5	1/2	1/2
26	6	5/6	29/30	29/30	13/30	1/5	1/5	1/2	1/2
27	6	7/30	7/30	5/6	1/30	3/5	3/5	1/2	1/2
28	7	4/7	4/7	3/7	1/7	1/2	1/2	1/2	2/3
29	7	2/7	4/7	2/7	2/7	1/2	1/2	1/3	1/2
30	7	1/7	5/7	1/7	1/7	1/2	1/2	1/3	1/2
31	8	1/2	3/4	3/4	1/2	1/3	1/3	2/3	1/2
32	8	2/5	4/5	1/2	2/5	1/2	3/5	1/2	2/3
33	8	3/5	1/2	1/5	1/5	1/3	1/2	1/2	2/3
34	8	2/3	1/2	1/3	1/4	1/2	1/2	2/3	1/2
35	8	3/8	11/24	3/8	5/24	1/2	1/2	1/2	1/3
36	8	1/4	11/20	11/20	3/20	1/2	1/3	1/2	2/3
37	8	3/4	9/20	7/20	7/20	3/5	1/2	2/3	1/2
38	8	1/20	3/4	7/20	1/20	1/2	1/3	2/3	1/2
39	8	7/8	7/24	1/8	1/24	1/2	1/2	2/3	1/2
40	8	3/4	3/4	1	0	2/3	2/3	1/2	2/3

Table 1 continued

♯	Length	θ_1	θ_2	θ_3	θ_4	σ_{12}	σ_{23}	σ_{13}	σ_{24}
41	8	3/4	3/20	3/20	1/20	1/2	3/5	1/3	1/2
42	9	8/15	8/15	11/15	7/15	4/5	2/5	2/5	2/5
43	9	8/15	1/15	14/15	1/15	4/5	4/5	2/5	1/3
44	9	3/5	3/5	2/3	2/5	3/5	1/5	2/5	2/5
45	9	1/3	4/5	4/5	1/5	4/5	1/5	2/3	3/5
46	9	4/15	4/15	4/15	2/15	1/5	1/5	2/5	1/3
47	9	2/15	2/15	13/15	1/15	3/5	1/5	2/5	2/5
48	10	3/5	1/2	1/2	1/5	2/3	1/2	1/2	1/2
49	10	9/10	17/30	17/30	13/30	1/2	1/2	1/2	2/5
50	10	3/5	3/5	7/10	3/10	1/2	2/3	1/2	1/2
51	10	1/3	2/5	1/3	1/3	1/2	1/2	1/5	1/2
52	10	11/30	11/30	11/30	3/10	1/2	1/2	1/2	1/5
53	10	2/3	4/5	2/3	1/3	1/2	1/2	3/5	1/2
54	10	29/30	7/10	29/30	1/30	1/2	1/2	1/2	1/5
55	10	9/10	23/30	23/30	7/30	1/2	1/2	2/5	1/2
56	10	1/5	1/5	9/10	1/10	1/2	2/3	1/2	1/2
57	10	3/5	2/5	3/5	2/5	2/5	2/5	1/3	3/5
58	10	1	1	1	3/5	0	1/5	1/3	1/3
59	10	4/5	4/5	4/5	4/5	1/5	0	1/3	1/3
60	10	1	1	1	1/5	3/5	3/5	3/5	1/3
61	12	1/2	1/2	1/2	1/3	1/2	3/4	2/3	1/3
62	12	1/2	2/3	2/3	1/2	1/4	1/2	1/2	1/2
63	12	1/2	2/3	3/5	2/5	2/5	2/3	1/2	1/2
64	12	3/5	1/2	1/3	1/5	1/2	1/2	3/5	1/2
65	12	1/2	4/5	4/5	1/3	2/3	1/5	1/2	1/2
66	12	7/12	7/12	29/60	19/60	3/5	1/3	1/2	1/2
67	12	37/60	31/60	41/60	13/60	1/2	1/2	1/2	2/5
68	12	7/10	17/30	3/10	7/30	2/5	3/5	4/5	1/2
69	12	5/12	7/12	5/12	5/12	1/2	3/4	2/3	1/3
70	12	7/12	53/60	43/60	7/12	1/3	1/5	1/2	1/2
71	12	1/3	4/5	3/5	1/3	3/5	1/2	3/5	4/5
72	12	11/12	11/12	37/60	13/60	1/5	2/3	1/2	1/2
73	12	29/30	9/10	9/10	11/30	2/5	1/2	2/5	1/5
74	12	17/60	11/60	59/60	7/60	1/2	1/2	2/5	1/2
75	12	1/12	11/12	11/60	1/60	2/5	2/3	1/2	1/2
76	12	1	1	5/6	1/6	1/2	3/4	1/2	1/2
77	12	11/12	11/12	11/12	1/12	1/4	1/2	1/3	2/3

Table 1 continued

♯	Length	θ_1	θ_2	θ_3	θ_4	σ_{12}	σ_{23}	σ_{13}	σ_{24}
78	15	8/15	7/15	4/5	7/15	2/5	2/5	1/2	1/2
79	15	13/15	1	1	7/15	1/5	0	1/2	1/2
80	15	1/3	3/5	3/5	1/3	3/5	0	1/2	1/2
81	15	1/3	1/3	3/5	1/3	3/5	3/5	1/2	1/2
82	15	4/15	4/15	2/5	4/15	1/5	1/5	1/2	1/2
83	15	14/15	14/15	3/5	1/15	1/5	4/5	1/2	1/2
84	15	4/5	2/3	1/3	1/3	1/5	4/5	1/2	1/2
85	15	4/5	2/3	1/3	1/5	1/5	1	1/2	1/2
86	15	14/15	4/15	0	0	0	3/5	1/2	1/2
87	15	13/15	1/5	2/15	2/15	3/5	2/5	1/2	1/2
88	16	3/4	1/2	1/2	1/2	2/3	1/3	1/2	1/3
89	16	5/8	5/8	5/8	3/8	1/3	2/3	1/3	1/2
90	16	7/8	7/8	7/8	1/8	1/3	2/3	1/3	1/2
91	18	29/42	23/42	23/42	19/42	1/2	1/3	2/3	2/7
92	18	31/42	31/42	23/42	11/42	1/2	4/7	4/7	2/3
93	18	4/7	3/7	3/7	1/3	2/3	1/3	2/7	1/2
94	18	17/42	17/42	17/42	5/42	1/2	3/7	3/7	1/3
95	18	1/42	41/42	17/42	1/42	6/7	1/7	1/2	3/7
96	18	2/3	2/3	2/3	2/3	0	1/5	3/5	3/5
97	18	2/7	2/7	1/3	2/7	1/7	1/7	1/2	4/7
98	18	6/7	6/7	6/7	1/3	1/2	3/7	3/7	1/3
99	18	1	1/3	0	0	1/5	1	2/5	2/5
100	18	29/42	29/42	13/42	11/42	1/7	6/7	1/2	3/7
101	18	37/42	37/42	37/42	1/42	1/2	1/3	1/3	5/7
102	20	3/5	1/2	1/2	2/5	4/5	2/5	1/2	1/3
103	20	1/2	1/2	4/5	1/5	2/3	3/5	1/2	3/5
104	20	3/5	1/2	1/3	1/3	1/2	1/3	1/2	3/5
105	20	1/2	1/3	4/5	1/3	1/2	3/5	1/3	1/2
106	20	13/20	13/20	29/60	11/60	1/2	1/2	2/3	3/5
107	20	43/60	37/60	11/20	11/20	1/2	2/3	1/2	2/5
108	20	17/20	19/60	3/20	1/60	1/2	3/5	1/2	2/3
109	20	1	1	7/10	3/10	1/3	3/5	1/2	2/5
110	20	1/20	47/60	7/60	1/20	2/3	1/2	2/5	1/2
111	20	1	9/10	9/10	0	1/5	2/5	2/3	1/2
112	24	1/2	1/3	2/3	1/3	1/2	2/5	1/2	1/3
113	24	7/12	5/12	7/12	1/4	2/5	1/2	1/3	1/2
114	24	3/4	1/12	1/12	1/12	1/2	3/5	1/3	1/2
115	30	1/2	1/2	2/5	1/3	1/2	1/5	1/2	1/2

Table 1 continued

♯	Length	θ_1	θ_2	θ_3	θ_4	σ_{12}	σ_{23}	σ_{13}	σ_{24}
116	30	1/2	1/2	2/3	1/5	1/2	2/5	1/2	1/2
117	30	8/15	19/30	19/30	7/15	1/5	1/2	1/2	1/2
118	30	11/15	11/15	17/30	13/30	1/2	3/5	1/2	1/2
119	30	1/15	23/30	7/30	1/15	2/5	1/2	1/2	1/2
120	30	13/15	2/15	1/30	1/30	1/2	4/5	1/2	1/2
121	36	1/2	2/3	2/3	1/2	1/3	3/5	1/2	2/5
122	36	1	1	5/6	1/6	3/5	1/3	1/2	2/5
123	40	1/2	1/2	1/2	2/5	1/2	4/5	2/3	1/3
124	40	1/2	4/5	1/2	1/2	2/5	2/3	2/5	1/2
125	40	11/20	9/20	9/20	9/20	1/2	1/5	1/3	2/3
126	40	13/20	13/20	13/20	7/20	2/3	3/5	3/5	1/2
127	40	17/20	17/20	17/20	3/20	1/2	1/3	3/5	2/5
128	40	19/20	19/20	19/20	1/20	1/2	4/5	1/3	2/3
129	72	1/2	1/2	1/2	1/3	4/5	1/2	2/5	3/5
130	72	5/12	7/12	5/12	5/12	1/2	4/5	3/5	2/5
131	72	1/12	1/12	11/12	1/12	1/2	1/5	2/5	3/5

Having all elements of the orbit, we select a subset with the following condition: for such k that $N_k = 1$ we require that $N_{k+1} = 2$ (observe that the index k is defined modulo n).

For the remaining tuples $\mathcal{M}^{(n)}$'s we allow only the following braid group actions: any $\mathcal{B}_{m,m+1}$ and $\mathcal{B}_{m+1,m}$ for $m \neq k-1, k, k+1$; compositions $\mathcal{B}_{k,k+1}\mathcal{B}_{k-1,k}$, $\mathcal{B}_{k+1,k}\mathcal{B}_{k+2,k+1}$, $\mathcal{B}_{k,k+1}\mathcal{B}_{k+1,k+2}$ and $\mathcal{B}_{k+1,k}\mathcal{B}_{k,k-1}$, where k is such value that $N_k = 1$.

These allowed braid group actions preserve the condition from the previous paragraph: $N_k = 1 \to N_{k+1} = 2$.

Next, we can do the reduction of elements of this set, joining the matrices $M_k,\ M_{k+1}$ for which $N_k = 1,\ N_{k+1} = 2$ into their product $M_k \cdot M_{k+1}$.

To describe such reduction more simply we use the fact that the set is closed under cyclical permutation (see 3), and define the reduction only for such $\mathcal{M}^{(n)}$'s that $N_{n-1} = 1$, $N_n = 2$. The reduction operation will be notated as $r_{k,k+1}$,

$$r_{k,k+1}\{...M_k,\ M_{k+1}, ...\} \to \{...M_k \cdot M_{k+1}, ...\},$$
$$r_{n,1}\{M_1, ...M_n\} \to \{M_n \cdot M_1, ...\},$$

so we have:

$$\begin{aligned}
&\mathcal{B}_{m,m+1}\, r_{n-1,n}\mathcal{M}^{(n)} = r_{n-1,n}\mathcal{B}_{m,m+1}\mathcal{M}^{(n)}, \quad m < n-2,\\
&\mathcal{B}_{m+1,m}\, r_{n-1,n}\mathcal{M}^{(n)} = r_{n-1,n}\mathcal{B}_{m+1,m}\mathcal{M}^{(n)}, \quad m < n-2,\\
&\mathcal{B}_{n-2,n-1}\, r_{n-1,n}\mathcal{M}^{(n)} = r_{n-2,n-1}\mathcal{B}_{n-1,n}\mathcal{B}_{n-2,n-1}\mathcal{M}^{(n)},\\
&\mathcal{B}_{n-1,n-2}\, r_{n-2,n-1}\mathcal{M}^{(n)} = r_{n-1,n}\mathcal{B}_{n-1,n-2}\mathcal{B}_{n,n-1}\mathcal{M}^{(n)},
\end{aligned}$$

$$\mathcal{B}_{n-1,1}\, r_{n,1}\mathcal{M}^{(n)} = r_{n-1,n}\mathcal{B}_{n,1}\mathcal{B}_{n-1,n}\mathcal{M}^{(n)},$$
$$\mathcal{B}_{1,n-1}\, r_{n-1,n}\mathcal{M}^{(n)} = r_{n,1}\mathcal{B}_{n,n-1}\mathcal{B}_{1,n}\mathcal{M}^{(n)},$$

and for each obtained $\mathcal{M}^{(n-1)}$ add all its copies obtained by cyclical permutation.

Now we have a set of the tuples $\mathcal{M}^{(n-1)}$'s, finite and closed under all braid group actions.

It is for sure either a finite orbit or a set of several finite orbits.

Lemma is proven. □

Lemma 3 *As a corollary of the previous lemma, every* ${\mathcal{M}_F}^{(n)}$ *can be constructed from two* ${\mathcal{M}_F}^{(n-1)}$*'s:*

Proof We take any two tuples ${\mathcal{M}_F}^{(n-1)}$, which coincide with all matrices, except for two neighboring ones. In one ${\mathcal{M}_F}^{(n-1)}$, we call these two matrices A and B

$$A,\ B,\ M_3 \dots M_{n-1}$$

and in another tuple we call them C and $C^{-1} A\, B$ (so that the product of both matrices is equal to the one in the first tuple)

$$C,\ C^{-1}AB,\ M_3 \dots M_{n-1}.$$

We construct from them the tuple $\mathcal{M}^{(n)}$:

$$C,\ C^{-1}A,\ B,\ M_3 \dots M_{n-1}.$$

However, we are not yet sure that it belongs to a finite orbit; nevertheless every tuple ${\mathcal{M}_F}^{(n)}$ can be constructed in such way. □

Using all pairs of ${\mathcal{M}_F}^{(n-1)}$'s we can get a list of the tuples $\mathcal{M}^{(n)}$'s which is a complete list of the candidates for ${\mathcal{M}_F}^{(n)}$'s. Then we check which of them really generate finite orbits. It gives us a hope to get the complete list of ${\mathcal{M}_F}^{(n)}$'s by a finite procedure.

For $n = 1,\ 2,\ 3$ the problem of classification of finite orbits is trivial: every $\mathcal{M}^{(1)}$, $\mathcal{M}^{(2)}$ or $\mathcal{M}^{(3)}$ generates a finite orbit of length 1.

For $n = 4$ this problem was solved in our paper about Painleve-VI equation [3].

Therefore, using the list of finite orbits of $\mathcal{M}^{(4)}$'s, we can obtain the list of finite orbits of $\mathcal{M}^{(5)}$'s, and then using the list of $\mathcal{M}^{(5)}$'s, obtain the corresponding list for $\mathcal{M}^{(6)}$'s et cetera.

In this paper we developed the algorithm for exact and exhaustive search of $\mathcal{M}^{(n)}$ tuples. Our aim was to make this algorithm in such a way that it needs only simple arithmetic and algebra. This allowed us designing a special computer program to perform this search for $n = 5$, because it would be too many calculation for a human.

5 Example of construction of $\mathcal{M}_F{}^{(5)}$ with direct using of matrices

Let us assume that in the list below the following five $\mathcal{M}_F{}^{(4)}$'s occur. Let us call them A, B, C, D, E:

$$\begin{aligned}
A:\ &[A_1,\quad A_2,\quad A_3,\quad A_4]\\
&=\left[\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \begin{pmatrix}1&1\\-1&0\end{pmatrix},\begin{pmatrix}0&-i\\-i&1\end{pmatrix},\ \begin{pmatrix}i&i-1\\0&-i\end{pmatrix}\right];\\
B:\ &[B_1,\quad B_2,\quad B_3,\quad B_4]\\
&=\left[\begin{pmatrix}1&1\\-1&0\end{pmatrix},\ \begin{pmatrix}1&0\\0&1\end{pmatrix},\ \begin{pmatrix}0&-i\\-i&1\end{pmatrix},\ \begin{pmatrix}i&i-1\\0&-i\end{pmatrix}\right];\\
C:\ &[C_1,\quad C_2,\quad C_3,\quad C_4]\\
&=\left[\begin{pmatrix}1&1\\-1&0\end{pmatrix},\ \begin{pmatrix}0&-1\\1&1\end{pmatrix},\ \begin{pmatrix}-i&1-i\\0&i\end{pmatrix},\ \begin{pmatrix}i&i-1\\0&-i\end{pmatrix}\right];\\
D:\ &[D_1,\quad D_2,\quad D_3,\quad D_4]\\
&=\left[\begin{pmatrix}1&1\\-1&0\end{pmatrix},\ \begin{pmatrix}0&-1\\1&1\end{pmatrix},\ \begin{pmatrix}1&1\\-1&0\end{pmatrix},\ \begin{pmatrix}0&-1\\1&1\end{pmatrix}\right];\\
E:\ &[E_1,\quad E_2,\quad E_3,\quad E_4]\\
&=\left[\begin{pmatrix}0&-1\\1&1\end{pmatrix},\ \begin{pmatrix}1&1\\-1&0\end{pmatrix},\ \begin{pmatrix}0&-i\\-i&1\end{pmatrix},\ \begin{pmatrix}1&i\\i&0\end{pmatrix}\right].
\end{aligned}$$

In fact, A and B belong to one orbit of the length 1 (orbit number 2 in the Table 1), C belongs to an orbit of the length 6 (a symmetry of the orbit number 15), D belongs to another orbit of the length 1 (orbit number 1, all matrices in the tuple D can be diagonalized simultaneously), and E—to the orbit of length 3 (a symmetry of the orbit number 5).

We will obtain $\mathcal{M}^{(5)}$ from these tuples by the procedure given below.

Due to the construction principle exposed in the previous chapter, we notice that $A_3 = B_3$ and $A_4 = B_4$, so the tuples A and B differ only by two first matrices.

Then we can perform *induction* of the tuple A into tuple F, meaning that A_1 is $F_1 \cdot F_2$, renaming A_2 to F_3, A_3 to F_4 and A_4 to F_5:

$$[A_1,\quad A_2,\quad A_3,\quad A_4] = [F_1 \cdot F_2,\quad F_3,\quad F_4,\quad F_5].$$

But it doesn't give us information about F_1 and F_2. To obtain it we can use information from the tuple B:

$$[B_1,\quad B_2,\quad B_3,\quad B_4] = [F_1,\quad F_2 \cdot F_3,\quad F_4,\quad F_5].$$

In such a way we have found all the matrices F:

$$\left[\begin{pmatrix}1&1\\-1&0\end{pmatrix},\ \begin{pmatrix}0&-1\\1&1\end{pmatrix},\ \begin{pmatrix}1&1\\-1&0\end{pmatrix},\ \begin{pmatrix}0&-i\\-i&1\end{pmatrix},\ \begin{pmatrix}i&i-1\\0&-i\end{pmatrix}\right]. \tag{13}$$

But we can also use C, D and E for the next *inductions*:

$$\begin{aligned}
[C_1, \quad C_2, \quad C_3, \quad C_4] &= [F_1, \quad F_2, \quad F_3 \cdot F_4, \quad F_5],\\
[D_1, \quad D_2, \quad D_3, \quad D_4] &= [F_1, \quad F_2, \quad F_3, \quad F_4 \cdot F_5],\\
[E_1, \quad E_2, \quad E_3, \quad E_4] &= [F_2, \quad F_3, \quad F_4, \quad F_5 \cdot F_1].
\end{aligned}$$

Therefore despite using the pair A and B we could use similarly any of the pairs B and C, C and D, D and E or E and A.

Observe that for the signature formalism which will be described in the next chapter, we will need all the above written five 4-tuples of matrices.

The previous formulae still do not prove that F generates a finite orbit. But due to the Lemma 3 we are sure that every ${\mathcal{M}_F}^{(5)}$ can be constructed in a similar way.

For the F case we can check explicitly whether the generated orbit is finite or infinite. It turns out to be a finite orbit of length 16 (orbit number 9 in the Table 9), further it will be called a tetrahedral type orbit.

In order to obtain an exhaustive list of the tuples ${\mathcal{M}_F}^{(5)}$ we must repeat the procedure described above for each five ${\mathcal{M}_F}^{(4)}$ from the list of all possible ${\mathcal{M}_F}^{(4)}$'s.

The method of classification of finite $\mathcal{M}^{(5)}$ orbits, described in this example, cannot be used in practice, because the set of ${\mathcal{M}_F}^{(4)}$ is infinite.

The set of ${\mathcal{M}_F}^{(4)}$ can be described as a finite list using free parameters. Now we will repeat the procedure of this example, using ${\mathcal{M}_F}^{(4)}$'s in a parametric form as a source of our construction.

First, assume that written below tuple ${\mathcal{M}_F}^{(4)}$ with three parameters occurs in the list. We will call it A, and we will use it as the first stage of construction of F tuple:

$$\begin{aligned}
A: \quad & [A_1, \quad A_2, \quad A_3, \quad A_4]\\
= \quad & [F_1 \cdot F_2, \quad F_3, \quad F_4, \quad F_5]\\
= & \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2\cos(\pi x) & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\exp(i\pi z) \\ \exp(-i\pi z) & 2\cos(\pi y) \end{pmatrix}, \right.\\
& \left. \begin{pmatrix} \exp(i\pi z) & 2\exp(i\pi z)\cos(\pi x) - 2\cos(\pi y) \\ 0 & \exp(-i\pi z)) \end{pmatrix} \right].
\end{aligned}$$

Next, we find in the list another tuple, also three-parameteric:

$$\left[\begin{pmatrix} 2\cos(\pi x) & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\exp(i\pi z) \\ \exp(-i\pi z) & 2\cos(\pi y) \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} \exp(i\pi z) & 2\exp(i\pi z)\cos(\pi x) - 2\cos(\pi y) \\ 0 & \exp(-i\pi z)) \end{pmatrix} \right].$$

We are going to call it B, but we must rename the parameters in it, to avoid collision of our notations:

$$
\begin{aligned}
B: &\quad [B_1, \quad B_2, \quad B_3, \quad B_4] \\
&= \left[\begin{pmatrix} 2\cos(\pi a) & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -\exp(i\pi c) \\ \exp(-i\pi c) & 2\cos(\pi b) \end{pmatrix}, \right. \\
&\quad \left. \begin{pmatrix} \exp(i\pi c) & 2\exp(i\pi c)\cos(\pi a) - 2\cos(\pi b) \\ 0 & \exp(-i\pi c)) \end{pmatrix} \right].
\end{aligned}
$$

Now we have to find the condition on the parameters that lead to the equalities $B_3 = F_4$ and $B_4 = F_5$ up to common conjugation.

It gives us a discrete set of possibilities:

$$
a = \pm x, \quad b = \pm y, \quad c = \pm z,
$$

and we will choose one of them:

$$
a = -x, \quad b = -y, \quad c = -z.
$$

Therefore the tuple B turns into the following expression:

$$
\begin{aligned}
B: &\quad [B_1, \quad B_2, \quad B_3, \quad B_4] \\
&= \left[\begin{pmatrix} 2\cos(\pi x) & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -\exp(-i\pi z) \\ \exp(i\pi z) & 2\cos(\pi y) \end{pmatrix}, \right. \\
&\quad \left. \begin{pmatrix} \exp(-i\pi z) & 2\exp(-i\pi z)\cos(\pi x) - 2\cos(\pi y) \\ 0 & \exp(i\pi z)) \end{pmatrix} \right].
\end{aligned}
$$

Now in order to make B_3 to coincide with F_4 and B_4 with F_5 we will do a common conjugation

$$
\begin{aligned}
B_\nu \to &\begin{pmatrix} \exp(i\pi z)\cos(\pi x) - \cos(\pi y) & i\sin(\pi z) \\ -i\sin(\pi z) & \exp(-i\pi z)\cos(\pi x) - \cos(\pi y) \end{pmatrix} \cdot B_\nu \cdot \\
&\begin{pmatrix} \exp(i\pi z)\cos(\pi x) - \cos(\pi y) & i\sin(\pi z) \\ -i\sin(\pi z) & \exp(-i\pi z)\cos(\pi x) - \cos(\pi y) \end{pmatrix}^{-1},
\end{aligned}
$$

and obtain the following expression:

$$
\begin{aligned}
B: &\quad [B_1, \quad B_2, \quad B_3, \quad B_4] \\
&= \left[\begin{pmatrix} 2\cos(\pi x) & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -\exp(i\pi z) \\ \exp(-i\pi z) & 2\cos(\pi y) \end{pmatrix}, \right. \\
&\quad \left. \begin{pmatrix} \exp(i\pi z) & 2\exp(i\pi z)\cos(\pi x) - 2\cos(\pi y) \\ 0 & \exp(-i\pi z)) \end{pmatrix} \right].
\end{aligned}
$$

Finally, we can put $F_1 = B_1$ and $F_2 \cdot F_3 = B_2$, that is why $F_2 = B_2 \cdot F_3^{-1}$, and we obtain

$$
\begin{aligned}
F:\quad &[F_1,\quad F_2,\quad F_3,\quad F_4,\quad F_5]\\
&=\left[\begin{pmatrix} 2\cos(\pi x) & 1\\ -1 & 0\end{pmatrix},\quad \begin{pmatrix} 0 & -1\\ 1 & 2\cos(\pi x)\end{pmatrix},\right.\\
&\begin{pmatrix} 2\cos(\pi x) & 1\\ -1 & 0\end{pmatrix},\quad \begin{pmatrix} 0 & -\exp(i\pi z)\\ \exp(-i\pi z) & 2\cos(\pi y)\end{pmatrix},\\
&\left.\begin{pmatrix} \exp(i\pi z) & 2\exp(i\pi z)\cos(\pi x)-2\cos(\pi y)\\ 0 & \exp(-i\pi z))\end{pmatrix}\right].
\end{aligned}
$$

On the next step, we will get from the list one more tuple $\mathcal{M}_F{}^{(4)}$ and call it C:

$$
\begin{aligned}
C:\quad &[C_1,\quad C_2,\quad C_3,\quad C_4]\\
&=\left[\begin{pmatrix} 1 & 1\\ -1 & 0\end{pmatrix},\quad \begin{pmatrix} 0 & -1\\ 1 & 1\end{pmatrix},\quad \begin{pmatrix} -i & 1-i\\ 0 & i\end{pmatrix},\quad \begin{pmatrix} i & i-1\\ 0 & -i\end{pmatrix}\right],
\end{aligned}
$$

and will require the conditions $C_1 = F_1, C_2 = F_2, C_3 = F_3 \cdot F_4, C_4 = F_5$. To achieve this we don't need any common conjugation and must fix only all the parameters:

$$x = 1/3,\quad y = 1/3,\quad z = 1/2.$$

Therefore in this way we obtain:

$$
\begin{aligned}
F:\quad &[F_1,\quad F_2,\quad F_3,\quad F_4,\quad F_5] =\\
&=\left[\begin{pmatrix} 1 & 1\\ -1 & 0\end{pmatrix},\quad \begin{pmatrix} 0 & -1\\ 1 & 1\end{pmatrix},\quad \begin{pmatrix} 1 & 1\\ -1 & 0\end{pmatrix},\quad \begin{pmatrix} 0 & -i\\ -i & 1\end{pmatrix},\quad \begin{pmatrix} i & i-1\\ 0 & -i\end{pmatrix}\right].
\end{aligned}
$$

Further on, we take one more 4-tuple and call it D:

$$
\begin{aligned}
D:\quad &[D_1,\quad D_2,\quad D_3,\quad D_4]\\
&=\left[\begin{pmatrix} \exp(i\pi f) & 0\\ k & \exp(-i\pi f)\end{pmatrix},\quad \begin{pmatrix} \exp(i\pi g) & 0\\ l & \exp(-i\pi g)\end{pmatrix},\right.\\
&\left.\begin{pmatrix} \exp(i\pi h) & 0\\ m & \exp(-i\pi h)\end{pmatrix},\quad \begin{pmatrix} \exp(-if-ig-ih) & 0\\ n & \exp(if+ig+ih)\end{pmatrix}\right].
\end{aligned}
$$

Now we have to provide the equalities $D_1 = F_1$, $D_2 = F_2$, $D_3 = F_3$.

From the equality $Tr\ D_1 = Tr\ F_1$ up to a common conjugation we obtain $f = 1/3$. Further, from the equalitis $D_1 \cdot D_2 = F_1 \cdot F_2 = 1$ and $D_2 \cdot D_3 = F_2 \cdot F_3 = 1$ we get $g = -1/3, h = 1/3, l = -k, m = k, n = -k$. Therefore in this way we obtain

$$
\begin{aligned}
D:\quad &[D_1,\quad D_2,\quad D_3,\quad D_4]\\
&=\left[\begin{pmatrix} \frac{1+i\sqrt{3}}{2} & 0\\ k & \frac{1-i\sqrt{3}}{2}\end{pmatrix},\quad \begin{pmatrix} \frac{1-i\sqrt{3}}{2} & 0\\ -k & \frac{1+i\sqrt{3}}{2}\end{pmatrix},\right.\\
&\left.\begin{pmatrix} \frac{1+i\sqrt{3}}{2} & 0\\ k & \frac{1-i\sqrt{3}}{2}\end{pmatrix},\quad \begin{pmatrix} \frac{1-i\sqrt{3}}{2} & 0\\ -k & \frac{1+i\sqrt{3}}{2}\end{pmatrix}\right].
\end{aligned}
$$

Now in order to make D_1 and F_1 equal we perform a common conjugation:

$$D_\nu \to \begin{pmatrix} 0 & 1 \\ k & -\frac{1+i\sqrt{3}}{2} \end{pmatrix} \cdot D_\nu \cdot \begin{pmatrix} 0 & 1 \\ k & -\frac{1+i\sqrt{3}}{2} \end{pmatrix}^{-1}$$

and obtain

$$\begin{aligned} D: \quad & [D_1, \quad D_2, \quad D_3, \quad D_4] \\ = & \left[\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right]. \end{aligned}$$

In such a way the conditions $D_1 = F_1$, $D_2 = F_2$, $D_3 = F_3$, $D_4 = F_4 \cdot F_5$ are satisfied.

Finally we can find 4-tuple the written below call it E:

$$\begin{aligned} E: \quad & [E_1, \quad E_2, \quad E_3, \quad E_4] \\ = & \left[\begin{pmatrix} -U^2 & -U - U^{-1} \\ 0 & -U^{-2} \end{pmatrix}, \quad \begin{pmatrix} U & 1 \\ 0 & U^{-1} \end{pmatrix}, \quad \begin{pmatrix} 0 & U \\ -U^{-1} & 1 \end{pmatrix}, \right. \\ & \left. \begin{pmatrix} -U^{-3} & U^4 + U^2 + 1 + U^{-2} \\ -U^{-4} & U + U^{-1} + U^{-3} \end{pmatrix} \right], \end{aligned}$$

where parameter U can also be defined as $\exp(i\pi u)$.

In order to satisfy the condition $Tr\ E_2 = Tr\ F_2$ we put $U = \exp(i\pi/3)$ and obtain

$$\begin{aligned} E: \quad & [E_1, \quad E_2, \quad E_3, \quad E_4] \\ = & \left[\begin{pmatrix} \frac{1-i\sqrt{3}}{2} & -1 \\ 0 & \frac{1+i\sqrt{3}}{2} \end{pmatrix}, \quad \begin{pmatrix} \frac{1+i\sqrt{3}}{2} & 1 \\ 0 & \frac{1-i\sqrt{3}}{2} \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{1+i\sqrt{3}}{2} \\ \frac{-1+i\sqrt{3}}{2} & 1 \end{pmatrix}, \right. \\ & \left. \begin{pmatrix} 1 & \frac{-1-i\sqrt{3}}{2} \\ \frac{1-i\sqrt{3}}{2} & 0 \end{pmatrix} \right]. \end{aligned}$$

Now we do a common conjugation

$$E_\nu \to \begin{pmatrix} 1 + i\sqrt{3} & -i + \sqrt{3} \\ -2 & (1-i)(1-\sqrt{3}) \end{pmatrix} \cdot E_\nu \cdot \begin{pmatrix} 1 + i\sqrt{3} & -i + \sqrt{3} \\ -2 & (1-i)(1-\sqrt{3}) \end{pmatrix}^{-1}$$

and obtain the equality

$$\begin{aligned} E: \quad & [E_1, \quad E_2, \quad E_3, \quad E_4] \\ = & \left[\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ -i & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & i \\ i & 0 \end{pmatrix} \right]. \end{aligned}$$

Therefore $E_2 = F_2$, $E_3 = F_3$, $E_4 = F_4$ and $E_1 = F_5 \cdot F_1$.

Table 2 4-signatures for constructing of a 5-signature

	A	B	C	D	E
θ_1	0	1/3	1/3	1/3	1/3
θ_2	1/3	0	1/3	1/3	1/3
θ_3	1/3	1/3	1/2	1/3	1/3
θ_4	1/2	1/2	1/2	1/3	1/3
σ_{12}	1/3	1/3	0	0	0
σ_{23}	1/2	1/3	1/3	0	1/2
σ_{13}	1/3	1/2	2/3	2/3	1/3
σ_{24}	1/3	1/2	2/3	2/3	1/3

By a procedure described above the full list of ${\mathcal{M}_F}^{(5)}$ can be obtained with a finite number of steps, but it needs too much work to be made manually, and also requires rather complicated mathematics programming a computer.

In the next chapter we will describe how this work could be simplified using the signature formalism.

6 Example of construction of ${\mathcal{M}_F}^{(5)}$ with signature formalism

In this chapter we repeat the result of the previous chapter using the signature formalism.

To begin with, we perform it with explicit values only, without free parameters.

Assume that in the list of ${\mathcal{M}_F}^{(4)}$'s signatures the following five signatures, which will be called A, B, C, D, E, occur: see Table 2.

Now we must perform an induction in each of these signatures, transforming the 4-signature A into 5-signature A' and so on:

$$
\begin{aligned}
&[A_1, \quad A_2, \quad A_3, \quad A_4] \to [A'_1 \cdot A'_2, \quad A'_3, \quad A'_4, \quad A'_5],\\
&[B_1, \quad B_2, \quad B_3, \quad B_4] \to [B'_1, \quad B'_2 \cdot B'_3, \quad B'_4, \quad B'_5],\\
&[C_1, \quad C_2, \quad C_3, \quad C_4] \to [C'_1, \quad C'_2, \quad C'_3 \cdot C'_4, \quad C'_5],\\
&[D_1, \quad D_2, \quad D_3, \quad D_4] \to [D'_1, \quad D'_2, \quad D'_3, \quad D'_4 \cdot D'_5],\\
&[E_1, \quad E_2, \quad E_3, \quad E_4] \to [E'_2, \quad E'_3, \quad E'_4, \quad E'_5 \cdot E'_1].
\end{aligned}
$$

Therefore we must re-order the cells θ and σ in the new signatures A', B', C', D', E', which became particular signatures: see Table 3.

Now we must merge these five particular signatures. We will call the resulting signature by F. The merging process is possible because in each row there are the same values: see Table 4.

The reason for using five 4-signatures, not two of them, for constructing of 5-signature, is that some cells like σ_{13} or σ_{135} can be obtained from only one of the

Table 3 Particular signatures

	A'	B'	C'	D'	E'
θ_1		1/3	1/3	1/3	
θ_2			1/3	1/3	1/3
θ_3	1/3			1/3	1/3
θ_4	1/3	1/3			1/3
θ_5	1/2	1/2	1/2		
σ_{12}	0		0	0	
σ_{23}		0		0	0
σ_{34}	1/2		1/2		1/2
σ_{45}	1/3	1/3		1/3	
σ_{51}		1/3	1/3		1/3
σ_{13}				2/3	
σ_{24}					1/3
σ_{35}	1/3				
σ_{41}		1/2			
σ_{52}			2/3		
σ_{134}			2/3		
σ_{245}				2/3	
σ_{351}					1/3
σ_{412}	1/3				
σ_{523}		1/2			

signatures A, B, C, D, E. Due to this if we do not use all five of the 4-signatures we will not be able to construct the complete 5-signature.

Therefore we get the 5-signature F and, due to the Lemma 1, it can generate only one tuple of matrices. However, the existence of this one tuple is not guaranteed by the lemma. Neither it is guaranteed that this tuple generates a finite orbit.

Now we repeat the same procedure, starting from 4-signatures with parameters. These signatures are taken from the Table 1, possibly transformed by the symmetry group (3), (4), (5): A and B are taken from the orbit number 2 in the Table 1, C—from the orbit 15, D—from the orbit 1, and E—from the orbit 5. Some of these signatures are the members of orbits that are not represented in the Table 1 since the Table 1 represents only one member of each orbit. So we get the Table 5.

Now we perform an *induction* and obtain the following five particular 5-signatures: see Table 6.

For providing of equivalence of the values in every row modulo 2 and up to sign, we have a finite number of needed relations among the parameters which are also defined modulo 2.

We choose the same set of relations as in the previous chapter:

$$a = -x, \quad b = -y, \quad c = -z, \quad x = 1/3, \quad y = 1/3, \quad z = 1/2,$$

Table 4 Merging of the particular signatures

	A'	B'	C'	D'	E'	F
θ_1		1/3	1/3	1/3		1/3
θ_2			1/3	1/3	1/3	1/3
θ_3	1/3			1/3	1/3	1/3
θ_4	1/3	1/3			1/3	1/3
θ_5	1/2	1/2	1/2			1/2
σ_{12}	0		0	0		0
σ_{23}		0		0	0	0
σ_{34}	1/2		1/2		1/2	1/2
σ_{45}	1/3	1/3		1/3		1/3
σ_{51}		1/3	1/3		1/3	1/3
σ_{13}				2/3		2/3
σ_{24}					1/3	1/3
σ_{35}	1/3					1/3
σ_{41}		1/2				1/2
σ_{52}			2/3			2/3
σ_{134}			2/3			2/3
σ_{245}				2/3		2/3
σ_{351}					1/3	1/3
σ_{412}	1/3					1/3
σ_{523}		1/2				1/2

Table 5 4-signatures with parameters

	A	B	C	D	E
θ_1	0	a	1/3	f	$2u+1$
θ_2	x	0	1/3	g	u
θ_3	y	b	1/2	h	1/3
θ_4	z	c	1/2	$f+g+h$	u
σ_{12}	x	a	0	$f+g$	$3u+1$
σ_{23}	z	b	1/3	$g+h$	1/2
σ_{13}	y	c	2/3	$f+h$	1/3
σ_{24}	y	c	2/3	$f+h$	1/3

$$f = h = 1/3, \quad g = -1/3, \quad u = 1/3.$$

Then we can perform a merging process of particular signatures and obtain the signature F: see Table 7.

Table 6 Particular signatures with parameters

	A'	B'	C'	D'	E'
θ_1		a	1/3	f	
θ_2			1/3	g	u
θ_3	x			h	1/3
θ_4	y	b			u
θ_5	z	c	1/2		
σ_{12}	0		0	$f+g$	
σ_{23}		0		$g+h$	$3u+1$
σ_{34}	z		1/2		1/2
σ_{45}	x	a		$f+g+h$	
σ_{51}		b	1/3		$2u+1$
σ_{13}				$f+h$	
σ_{24}					1/3
σ_{35}	y				
σ_{41}		c			
σ_{52}			2/3		
σ_{134}			2/3		
σ_{245}				$f+h$	
σ_{351}					1/3
σ_{412}	y				
σ_{523}		c			

7 Constructing algorithm

In order to get the list of finite orbits of $\mathcal{M}^{(5)}$'s, we will need the following stages:

Stage 1. We take a list of signatures of $\mathcal{M}_F{}^{(4)}$ (including the signatures which correspond to triangular tuples).

We make this list to be finite using free parameters in some signatures. Further we allow the next form of cells of signatures with parameters: each cell can be equal to linear combination of several parameters, taken with integer coefficient, and the free term is a rational number with a denominator 2520 and determined modulo 2. Each parameter is also determined modulo 2.

Moreover, if there is a possibility of equivalent linear redefinition of the set of parameters with another set such that Jacobian of transformation of parameters is more than 1 by absolute value, then we do such the redefinition. E.g. if we have $\theta_1 = x + y, \theta_2 = x - y$ and $\theta_3 = 2x$, we replace it by $\theta_1 = z, \theta_2 = w$ and $\theta_3 = z + w$. The condition of non-existence of such redefinition will be called *minimal Jacobian condition*.

Each tuple $\mathcal{M}_F{}^{(4)}$ can be reproducted in 64 copies, using the symmetry transformations (3), (4) and (5).

Table 7 Merging with parameters

	A'	B'	C'	D'	E'	F
θ_1		– 1/3	1/3	1/3		1/3
θ_2			1/3	– 1/3	1/3	1/3
θ_3	1/3			1/3	1/3	1/3
θ_4	1/3	– 1/3			1/3	1/3
θ_5	1/2	– 1/2	1/2			1/2
σ_{12}	0		0	0		0
σ_{23}		0		0	2	0
σ_{34}	1/2		1/2		1/2	1/2
σ_{45}	1/3	– 1/3		1/3		1/3
σ_{51}		– 1/3	1/3		5/3	1/3
σ_{13}				2/3		2/3
σ_{24}					1/3	1/3
σ_{35}	1/3					1/3
σ_{41}		– 1/2				1/2
σ_{52}			2/3			2/3
σ_{134}			2/3			2/3
σ_{245}				2/3		2/3
σ_{351}					1/3	1/3
σ_{412}	1/3					1/3
σ_{523}		– 1/2				1/2

Stage 2. We make all possible 5-signatures, combining the 4-signatures in different ways, as in the example from the previous chapter. The set of such 5-signatures will be called $\mathcal{S}_5^C$.

Let us construct the signature, which we call $\mathcal{S}$, meaning that it is the signature of a $\mathcal{M}^{(5)}$, which consists of $M_1,\ M_2,\ M_3,\ M_4,\ M_5$ monodromy matrices.

First, we take any $\mathcal{M}_F{}^{(4)}$, defining it as $M_1M_2,\ M_3,\ M_4,\ M_5$. Knowing the signature of the tuple $\mathcal{M}_F{}^{(4)}$, we will have a particular signature of $\mathcal{M}^{(5)}$'s, containing eight numbers: θ_3, θ_4, θ_5, σ_{12}, σ_{34}, σ_{45}, σ_{35} and σ_{124}.

Let us take another $\mathcal{M}_F{}^{(4)}$, defining it as $M_1,\ M_2M_3,\ M_4,\ M_5$, and renaming the eight numbers of its signature into eight numbers of particular signature of $\mathcal{M}_5$: θ_1, $\theta_4, \theta_5, \sigma_{23}, \sigma_{45}, \sigma_{15}, \sigma_{14}$ and σ_{235}. Merging these two particular signatures of the same $\mathcal{M}_5$ into one particular signature, we demand the coincidence of two values of θ_4, two of θ_5 and two values of σ_{45}. As a result we obtain a particular signature of $\mathcal{M}^{(5)}$ with 13 defined cells.

If the particular signatures that are merging contain parameters, we must do the following procedure:

For each cell defined in both particular signatures and if at least in one of these cells contains parameters, we must consider two possibilities: 1) the values in these cells coincide exactly modulo 2, 2) the values coincide with a change of sign, also modulo 2. In total, we have two in the power number of such the cells possibilities.

For each of these possibilities we have a system of linear equations. To each cell corresponds one equation, and it is an equation defined modulo 2, in which variables are also defined modulo 2, all coefficients are integers, and the free term is a multiple of $1/2520$.

We solve this system by iterations:

Solving of the system

Step 0. On each step, we find the smallest in absolute value non-zero coefficient in the system. Let us call this coefficient k_{EV}. The equation in which it appears will be called E, and the variable by which it appears—V.

Step 1. If $k_{EV} = \pm 1$, we use the equation E to determine V via another variables. Then we come back to Step 0, but with smaller number of variables. Otherwise we go to Step 2.

Step 2. If all the coefficients are zero, then the process of solving of the system is almost finished. We check the free terms. If at least one free term is nonzero modulo 2, then this case produces no solutions. If all equations are satisfied, then the system is solved and the task is performed.

Step 3. If $|k_{EV}| \geq 2$, we look for the coefficient by the variable V in other equations, which is bigger or equal to k_{EV} by absolute value. If it exists, we add to each such equation the equation E with such a factor that coefficient by V becomes less than k_{EV} in absolute value. Then we come back to Step 0, having either smaller minimal coefficient or smaller number of nonzero coefficients. Otherwise we go to Step 4.

Step 4. We check if there is a coefficient in E which is bigger or equal to k_{EV} by absolute value. If it exists, we redefine the variable V by adding to it other variables with such factors that there remains no coefficient bigger or equal than k_{EV} in the equation E. Then we come back to Step 0, having either smaller minimal coefficient or smaller number of nonzero coefficients. Otherwise we go to Step 5.

Step 5. Here we have $|k_{EV}| \geq 2$, all other coefficients by V are zero, and all other coefficients in the equation E are zero too. We control that $|k_{EV}|$ is a divisor of 5040 and the free term of E is a multiple of $|k_{EV}|/2520$. If one of these conditions is violated, we treat this as a mistake of the algorithm. In fact, it never happens. Therefore we choose an integer value m from the interval $[0,\ |k_{EV}| - 1]$ (yet another branching of possibilities), divide the equation E by k_{EV} and add to its free term $2m/k_{EV}$. We have to do this because the equation is defined modulo 2. And we come back to Step 0. □

Solving the system, we perform the operations with variables (see Steps 1, 4, 5) simultaneously with parameters in signatures. When the system is solved, we get new particular signatures with new parameters (number of parameters can increase, decrease or even vanish).

In order to provide the *minimal Jacobian condition*, we transform the parameters in obtained particular signature with Gauss-like transformation.

Note that this procedure can give several results in case of particular signatures with parameters.

Next, we repeat this procedure thrice, merging the particular signature with particular signatures obtained by induction of $\mathcal{M}_F{}^{(4)}$'s, defined as $\{M_1,\ M_2,\ M_3 \cdot M_4,\ M_5\}$,

$\{M_1, M_2, M_3, M_4 \cdot M_5\}$ and $\{M_2, M_3, M_4, M_5 \cdot M_1\}$, in order to fill in all cells and obtain the complete signature $\mathcal{S}_5$, which is considered to be a signature of $\mathcal{M}^{(5)}$.

Using this procedure with all 4-signatures from the Table 1 (which are the signatures of all ${\mathcal{M}_F}^{(4)}$'s), we get a list of $\mathcal{S}_5$'s which we call ${\mathcal{S}^C}_5$. We know that the signature of every ${\mathcal{M}_F}^{(5)}$ belongs to this list.

Naively, we would check each signature ${\mathcal{S}^C}_5$ for *consistency*. The *inconsistent* ones we remove from the list. And for each consistent signature we reconstruct $\mathcal{M}^{(5)}$ as a tuple of matrices and check whether it generates a finite orbit.

But here there is a problem: the check of the *consistency* with the help of the computer program can not be done exactly and, moreover, for the tuple $\mathcal{M}_5$'s containing free parameters this task is impossible to perform with the help of computer algebra.

That is why we will proceed in another way.

Stage 3. The straightforward way to proceed would be the following sequence of steps:

Each signature from the list ${\mathcal{S}^C}_5$ we should check for consistency, and—if it is consistent—to reconstruct it into $\mathcal{M}^{(5)}$. Then, using these $\mathcal{M}^{(5)}$'s we should form the list and call it ${\mathcal{M}^C}_5$. Then on each element M of ${\mathcal{M}^C}_5$ we should act by ten braid group actions [see (1) and (2)] and check whether all the results of these actions also belong to ${\mathcal{M}^C}_5$. If they do not, than we would be sure that M generates an infinite orbit, and exclude it from the list ${\mathcal{M}^C}_5$. Then we should repeat this check until there remains nothing to exclude. After this, we would be sure that ${\mathcal{M}^C}_5$ contains only finite orbits, or possibly infinite orbits written by finite number of elements with parameters, so that during continuation of the orbit the parameters transform under an infinite group action.

But our wish is to avoid this complicated procedure of checking the consistency and reconstruction of tuples of matrices.

That's why we will use another approach. Starting from the list ${\mathcal{S}^C}_5$, we will treat it, directly considering the signatures and removing unwanted elements from this list, and the rest elements uniting into the orbits.

It is possible to determine the braid group actions onto the signature, and the result will be an *incomplete signature*.

Due to the definition of the incomplete signature, there are ten types of incomplete signatures, each containing sixteen cells: all five θ's, the $\sigma_{12}, \sigma_{23}, \sigma_{34}, \sigma_{45}, \sigma_{15}$ and six more σ's. Each type of the incomplete signatures is obtained from complete signatures by a specific braid group action. In more details, if we act on any signature by braid group action $\mathcal{B}_{a,b}$, then we get an incomplete signature, in which σ's containing index a and not containing b will be undefined.

If we have $\mathcal{M}^{(5)}$ which will be denoted M, and $S = \{\theta_1 ...\}$ is the signature of M, then the signature $S' = \{\theta'_1 ...\}$ of the result of braid group action $\mathcal{B}_{k,k+1}M$ will be

$$\begin{aligned} \theta'_m &= \theta_m, \quad m \neq k,\ k+1; \\ \theta'_k &= \theta_{k+1}, \quad \theta'_{k+1} = \theta_k; \\ \sigma'_{a,b...} &= \sigma_{a,b...}, \quad k,\ k+1 \notin \{a, b...\}; \\ {\sigma'}_{...k,k+1...} &= \sigma_{...k,k+1...}; \\ {\sigma'}_{...k...} &= \ ?; \end{aligned}$$

$$\sigma'_{...k+1...} = \sigma_{...k...}.$$

The symbol ? means that there is no simple way to calculate this cell, and we leave it undetermined.

Similarly, the signature $S'' = \{\theta''_1 ...\}$ of $\mathcal{B}_{k+1,k}M$ will be

$$\begin{aligned}
\theta''_m &= \theta_m, \quad m \neq k,\ k+1; \\
\theta''_k &= \theta_{k+1}, \quad \theta''_{k+1} = \theta_k; \\
\sigma''_{a,b...} &= \sigma_{a,b...}, \quad k,\ k+1 \notin \{a, b...\}; \\
\sigma''_{...k,k+1...} &= \sigma_{...k,k+1...}; \\
\sigma''_{...k...} &= \sigma_{...k+1...}; \\
\sigma''_{...k+1...} &= \ ?.
\end{aligned}$$

We remind that all indices here are modulo n.

Therefore our plan is the following:

1. We take any signature from $\mathcal{S}^C{}_5$ and call it S.
2. We act on S by all $2 \cdot n = 10$ braid group actions.
3. The result of each braid group action on S is an incomplete signature which we call S'.
4. If S does not contain independent parameters—we look in $\mathcal{S}^C{}_5$ for any signature which can be merged with S'. If there is no such the signature, we exclude the S from $\mathcal{S}^C{}_5$.
5. If S contains independent parameters, and so does S'—we look (in $\mathcal{S}^C{}_5$) for any signature which can be merged with S' without imposing conditions on parameters in S'.
6. If S contains independent parameters, and the step 5 fails, we look in $\mathcal{S}^C{}_5$ for any signature, let us call it S^*, which can be merged with S' after the imposing conditions on parameters in S'. For each such S^* (it can be one, more than one or none) we make a copy of S after imposing the same conditions on its parameters and add it to $\mathcal{S}^C{}_5$. After this we exclude S from $\mathcal{S}^C{}_5$.
7. We repeat this procedure for all members of $\mathcal{S}^C{}_5$ until it remains nothing to exclude.

□

Note that some signatures can be inconsistent, but since the braid group actions are natural only for consistent signatures—the inconsistent signatures are likely to be excluded by this procedure. However, some inconsistent signatures can remain in the list.

Note that we could write down the same procedure only for signatures without parameters, avoiding the steps 5 and 6, but it would be an infinite procedure with infinite list of signatures. Due to the steps 5 and 6 we obtain the same result by a finite sequence of steps.

To illustrate these procedure, we provide three examples [all gathered in the Table 8].

Table 8 Braiding of 5-signatures

	A	A'	A^*	B	B'	C	C'	C^*	$C' \wedge C^*$	$C_\times$
θ_1	1/3	1/3	1/3	1/2	1/2	y	y	b	0	0
θ_2	1/3	1/3	1/3	1/2	1/2	z	1/2	c	1/2	1/2
θ_3	1/3	1/3	1/3	1/2	1/2	1/2	z	1/2	1/2	1/2
θ_4	1/3	1/3	1/3	1/2	1/2	x	x	a	0	0
θ_5	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2
σ_{12}	0	2/3	2/3	1/3	3/5	$y+z$	1/2	$b+c$	1/2	1/2
σ_{23}	0	0	0	1/5	1/5	1/2	1/2	1/2	1/2	1/2
σ_{34}	1/2	1/3	1/3	1/5	4/5	1/2	$x-z$	1/2	1/2	1/2
σ_{45}	1/3	1/3	1/3	1/3	1/3	1/2	1/2	1/2	1/2	1/2
σ_{51}	1/3	1/3	1/3	2/5	2/5	1/2	1/2	1/2	1/2	1/2
σ_{13}	2/3		0	3/5		1/2		1/2	1/2	1/2
σ_{24}	1/3	1/2	1/2	2/3	1/5	$x+z$	1/2	$a+c$	1/2	1/2
σ_{35}	1/3		2/3	3/5		$x+y+z$		$a+b+c$	1/2	1/2
σ_{41}	1/2	1/2	1/2	1/2	1/2	$x+y$	$x+y$	$a+b$	0	0
σ_{52}	2/3	1/3	1/3	1/2	3/5	1/2	$x+y+z$	1/2	1/2	1/2
σ_{134}	2/3		1/3	3/5		1/2		1/2	1/2	1/2
σ_{245}	2/3	0	0	2/3	1/3	1/2	$y+z$	1/2	1/2	1/2
σ_{351}	1/3		1/2	4/5		$x-z$		$a-c$	1/2	1/2
σ_{412}	1/3	2/3	2/3	2/3	3/5	$x-y-z$	1/2	$a-b-c$	1/2	1/2
σ_{523}	1/2	1/2	1/2	3/5	3/5	$x-y$	$x-y$	$a-b$	0	0

First example is a 5-signature A without parameters, which is the signature of $\mathcal{M}_F{}^{(5)}$ from (13). We perform the braid group action $\mathcal{B}_{3,2}$ getting an incomplete signature A' and find in $\mathcal{S}^C{}_5$ the signature A^* which can be merged with A', so the Step 4 succeeds.

Second example is the signature B such that $B' = \mathcal{B}_{3,2}B$ can not be merged with any signature in $\mathcal{S}^C{}_5$.

In the third example we start from the signature C with parameters. For it, we get an incomplete signature $C' = \mathcal{B}_{3,2}C$. In order to merge C' with any member of $\mathcal{S}^C{}_5$, which will be called C^*, we take an element of $\mathcal{S}^C{}_5$, same as C, of course with other notations for its parameters: a, b, c replacing x, y, z. Further, during merging of C' and C^* we require some conditions on the parameters, and one of the possibilities for such conditions is $a = b = x = y = 0$, $c = z = 1/2$. The result of merging is called $C' \wedge C^*$ and the copy of C which is added to $\mathcal{S}^C{}_5$ after excluding C is called $C_\times$.

Stage 4. After all exclusions, the list $\mathcal{S}^C{}_5$ will be renamed as $\mathcal{S}^F{}_5$.

Therefore $\mathcal{S}^F{}_5$ is the list of signatures, closed under the braid group actions. That is why $\mathcal{S}^F{}_5$ is split under braid group actions into a number of pieces. But these pieces still are not the orbits.

Naively, we would do two steps to finally construct the orbits:

1. First, we must check consistency of all these signatures. If one signature is inconsistent, then all signatures from the piece, associated with it by braid group actions, are inconsistent too. After excluding inconsistent signatures, only the consistent ones remain.
   Now we can transform this list of signatures into list of tuples $\mathcal{M}^{(5)}$'s.
2. For each tuple $\mathcal{M}^{(5)}$ from this list we have to construct an orbit to make sure that it is a finite orbit. If $\mathcal{M}^{(5)}$ contains no parameters, the orbit must be finite because the number of $\mathcal{M}^{(5)}$'s without parameters in the list is finite. As for any $\mathcal{M}^{(5)}$ with parameters, if the procedure of construction of the orbit does not terminate for too many steps, we will try to write this orbit with a finite set of elements and introduce a group of transformation of the parameters.
   In fact, we are sure that every tuple $\mathcal{M}^{(5)}$ with one parameter generates a finite orbit, because the order of the group of transformations of one parameter cannot be bigger than 10080: transformations of the parameters must be linear, with integer coefficient, invertible, that's why the coefficient can be $\pm$ 1 only, and the free term must be a multiple of $1/2520$ and defined by modulo 2.
   Therefore, the decision about "construction of the orbit does not terminate for too many steps" we must do only for the orbits with two or more parameters.

In order to simplify calculations, we perform another procedure with the list ${\mathcal{S}^F}_5$ to obtain the same result:

We construct several classes of the tuples ${\mathcal{M}_F}^{(5)}$'s, which can be described simply as follows:

1. All ${\mathcal{M}_F}^{(4)}$'s plus the unit matrix.
   2-6. All matrices in a ${\mathcal{M}_F}^{(5)}$ belong to a finite subgroup of $SU(2)$ group. There are five such subgroups:
2. Cyclical group. All the matrices are diagonal.
3. Dihedral group. Every matrix is either diagonal or its diagonal elements are zeros.
4. Tetrahedral group.
5. Octahedral group.
6. Icosahedral group.

For all these tuples ${\mathcal{M}_F}^{(5)}$ we make 5-signatures and find the same signatures in list ${\mathcal{S}^F}_5$. For the types 1, 4, 5, 6 we can do it straightforwardly. As for types 2, 3 there exists an infinite number of cyclical and dihedral groups, but belonging of the signature to one of these groups can be checked by simple calculations.

All this can be done using a specially designed computer program.

The members of the list ${\mathcal{S}^F}_5$ which are not of the types 1, 2, 3, 4, 5, 6, we will consider manually.

8 Computer realization

At this stage, the classification is formulated in such a way that it can be carried out with a specially designed program.

The computer program for the algorithm from the previous chapter was written in the C++ language and ran on a personal computer for about 12 h.

To have confidence in accuracy of the calculations, we restricted the use of floating point numbers: double and complex variables are used only for constructing the list of 4-signatures and for constructing of tetrahedral, octahedral and icosahedral groups (for which the correct results are well known), for approximate checking of consistency of signatures without parameters (which result is only informative and does not influence further calculations) and also for the visualization of the progress bar.

In the program there is a variable called "errorcode". Normally it is zero, but in every abnormal situation it is assigned the code of the situation, and can never become zero again. The fact that this variable remains zero till the program finishes makes us believe that the program works correctly.

The values of the coefficients of the parameters in signature cells never exceed 4, so we are not afraid of arithmetical overflow.

Free terms in the cells are standard fractions with denominator 2520 and numerator an integer value which does not exceed 2520 by absolute value.

The source code for this program is available by request.

9 The result of computer calculations

The result of computer calculations is the list $\mathcal{S}^F{}_5$ which consists of 231 orbits of signatures (we call an *orbit of signatures* the subset of $\mathcal{S}^F{}_5$ connected with braid group actions).

From these 231, 128 were recognized as constructed from $\mathcal{M}^{(4)}$ by addition of unit matrix.

And from the remaining 103, 3, 19 and 71 were recognized respectively as tetrahedral, octahedral and icosahedral type (in Table 9, see Sect. 10, they are numbered respectively as 9–11, 12–30 and 31–101).

There remain ten orbits of signatures: one with four parameters, two with three parameters, two with one parameter and five without parameters.

They are presented in the Table 10, where each orbit is represented by one signature.

The orbit with number 0 with four parameters cooresponds to the *triangular* $\mathcal{M}^{(5)}$'s, thus it is outside the scope this paper and will be not considered below. For this type of orbits we can say also that it includes the case when all matrices belong to a cyclical subgroup of $SU(2)$ (i.e. when all matrices are diagonal), but it also includes other subtypes: when all matrices cannot be diagonalized simultaneously, but are lower triangular and some of them have nonzero [2, 1] element. Nevertheless, the signatures of $\mathcal{M}^{(n)}$'s of orbits belonging to these other subtypes are the same as signatures of $\mathcal{M}^{(n)}$ where all matrices belong to a cyclical group.

The tuple $\mathcal{M}^{(5)}$ for the orbit 0 is constructed in (14):

$$
\begin{aligned}
&M_1 = \begin{pmatrix} X & 0 \\ V_x & X^{-1} \end{pmatrix}, \quad M_2 = \begin{pmatrix} Y & 0 \\ V_y & Y^{-1} \end{pmatrix}, \quad M_3 = \begin{pmatrix} Z & 0 \\ V_z & Z^{-1} \end{pmatrix}, \\
&M_4 = \begin{pmatrix} W & 0 \\ V_w & W^{-1} \end{pmatrix}, \quad M_5 = \begin{pmatrix} (X\,Y\,Z\,W)^{-1} & 0 \\ V_5 & X\,Y\,Z\,W \end{pmatrix}
\end{aligned}
\tag{14}
$$

Table 9 List of 5-signatures which generate finite orbits

♯	Length	θ_1	θ_2	θ_3	θ_4	θ_5	σ_{12}	σ_{23}	σ_{34}	σ_{45}	σ_{51}	σ_{13}	σ_{24}
1	4	1/2	x	y	z	1/2	1/2	$x+y$	$y+z$	1/2	$x+y+z$	1/2	$x+z$
2		1/2	1/2	x	1/2	1/2	$x+y$	1/2	1/2	y	z	1/2	$x+z$
3	9	x	x	x	x	2/3	$2x$	$2x$	1/3	$3x$	1/2	$2x$	1/3
4	12	x	x	$2x+1$	x	x	1/3	1/2	1/2	1/3	$2x$	$3x+1$	$2x$
5	105	2/7	2/7	2/7	2/7	2/7	1/3	1/7	1/7	1/7	1/3	1/3	1/2
6	105	4/7	4/7	4/7	4/7	4/7	1/3	5/7	5/7	5/7	1/3	1/3	1/2
7	105	6/7	6/7	6/7	6/7	6/7	1/3	3/7	3/7	3/7	1/3	1/3	1/2
8	192	1	1	1	1	1	1/3	1/3	1/3	1/3	1/3	1/5	1/5
9	16	1/3	1/3	1/3	1/3	1/2	0	0	1/2	1/3	1/3	2/3	1/3
10	24	1/3	1/3	1/3	1/2	1/2	1/3	1/3	1/3	0	1/3	1/3	1/3
11	36	1/3	1/3	1/2	1/2	1/2	0	1/3	1/2	1/2	1/3	2/3	1/3
12	18	1/4	1/4	1/3	1/2	1/2	1/3	1/4	1/3	0	1/4	1/4	1/2
13	24	1/4	1/4	1/4	1/2	1/2	0	1/3	1/2	1/4	1/4	1/3	1/3
14	24	1/4	1/4	1/3	1/3	1/2	0	1/4	1/2	1/3	1/4	1/2	1/4
15	24	1/4	1/4	1/3	1/3	2/3	0	1/4	2/3	1/3	1/2	1/2	1/4
16	27	1/4	1/4	1/4	1/4	1/3	0	0	1/3	1/4	1/4	1/2	1/3
17	36	1/4	1/3	1/2	1/2	1/2	1/4	1/4	1/4	1/2	1/2	2/3	2/3
18	36	1/4	1/4	1/4	1/3	1/2	0	0	1/2	1/4	1/3	1/2	1/4
19	40	1/4	1/4	1/3	1/3	1/3	0	1/4	1/3	1/3	1/4	1/2	1/4
20	48	1/4	1/4	1/2	1/2	1/2	0	1/3	1/2	1/2	1/3	2/3	1/2
21	48	1/4	1/3	1/3	1/2	1/2	1/4	0	1/4	1/4	1/2	1/2	3/4
22	64	1/4	1/3	1/3	1/3	1/2	1/4	0	1/3	1/4	1/3	1/2	1/2
23	72	1/3	1/2	1/2	1/2	1/2	1/4	1/4	1/4	1/2	2/3	2/3	2/3
24	72	1/4	1/4	1/3	1/2	1/2	0	1/4	1/2	1/3	1/3	1/2	1/3
25	96	1/4	1/2	1/2	1/2	1/2	1/4	1/4	1/3	1/3	2/3	2/3	1/2
26	96	1/3	1/3	1/2	1/2	1/2	0	1/4	1/2	1/2	1/3	3/4	1/2
27	120	1/3	1/3	1/3	1/2	1/2	0	1/3	1/2	1/3	1/4	1/2	1/4
28	144	1/4	1/3	1/2	1/2	1/2	1/4	1/4	1/3	1/3	1/3	1/2	1/2
29	192	1/2	1/2	1/2	1/2	1/2	0	1/4	1/2	1/2	1/3	3/4	1/3
30	216	1/3	1/2	1/2	1/2	1/2	1/4	0	1/3	1/3	1/2	3/4	2/3
31	30	1/5	1/5	2/5	2/5	2/5	1/5	1/5	1/3	1/3	1/3	1/2	1/2
32	30	1/5	1/5	1/5	2/5	3/5	0	1/5	3/5	1/5	1/2	1/3	1/3
33	36	1/5	1/3	2/5	2/5	3/5	1/5	1/5	4/5	1/5	1/2	1/2	1/5

Table 9 continued

♯	Length	θ_1	θ_2	θ_3	θ_4	θ_5	σ_{12}	σ_{23}	σ_{34}	σ_{45}	σ_{51}	σ_{13}	σ_{24}
34	36	1/5	1/5	1/5	1/3	3/5	1/5	1/5	2/5	1/3	1/2	1/3	2/5
35	40	1/5	1/5	2/5	2/5	3/5	0	1/5	3/5	2/5	1/2	3/5	1/3
36	40	1/5	1/5	1/5	2/5	2/5	1/5	1/5	1/5	0	1/3	1/5	1/2
37	45	1/5	2/5	2/5	2/5	3/5	1/3	0	3/5	1/5	2/5	1/2	1/3
38	45	1/3	1/3	2/5	2/5	2/5	1/5	1/5	1/3	1/3	1/5	1/2	1/2
39	45	1/5	1/5	1/3	2/5	3/5	1/5	1/5	2/5	2/5	1/2	1/2	1/2
40	45	1/5	1/5	1/5	1/5	2/5	0	1/5	2/5	1/5	1/3	1/3	1/5
41	45	1/5	1/5	1/5	1/3	2/3	1/5	1/5	1/2	1/3	1/2	1/3	1/3
42	64	2/5	2/5	2/5	2/5	1/2	1/3	1/3	1/3	1/5	1/5	1/3	1/3
43	64	1/5	1/5	1/5	1/5	1/2	1/5	1/5	1/3	1/3	1/3	1/3	1/5
44	72	1/5	1/5	1/3	2/5	2/5	0	1/5	2/5	1/3	1/3	1/2	1/3
45	80	1/5	1/5	2/5	2/5	1/2	1/5	1/5	1/3	1/3	1/3	1/2	1/2
46	81	1/5	1/3	2/5	2/5	2/5	1/5	1/5	1/3	1/3	1/3	1/2	2/5
47	81	1/5	1/5	1/5	1/3	2/5	0	0	2/5	1/5	1/3	2/5	1/3
48	84	1/5	1/3	1/3	2/5	2/5	1/5	1/5	1/5	1/3	1/2	1/2	2/3
49	84	1/5	1/5	1/3	1/3	3/5	0	1/5	3/5	1/3	2/5	1/2	1/5
50	96	1/5	2/5	2/5	2/5	1/2	1/5	0	3/5	1/5	2/5	3/5	1/3
51	96	1/5	1/5	1/5	2/5	1/2	0	0	1/2	1/5	2/5	2/5	1/3
52	96	2/5	2/5	2/5	2/5	3/5	0	0	3/5	2/5	2/5	4/5	1/3
53	96	1/5	1/5	1/5	1/5	1/5	0	0	1/5	1/5	1/5	2/5	1/3
54	105	1/3	1/3	1/3	2/5	3/5	1/5	1/5	2/5	2/5	1/2	1/2	2/5
55	105	1/5	1/3	1/3	2/5	3/5	1/5	0	2/3	1/5	2/5	1/2	1/5
56	105	1/5	1/5	1/3	1/3	2/5	1/5	1/5	1/5	2/5	1/3	1/2	1/2
57	105	1/5	1/5	1/3	1/3	2/3	0	1/5	2/3	1/3	1/2	1/2	1/5
58	108	1/3	2/5	2/5	2/5	2/5	1/5	0	1/3	1/3	2/5	2/3	3/5
59	108	1/5	1/5	1/5	1/5	1/3	0	0	1/3	1/5	1/5	2/5	1/5
60	120	1/5	1/3	1/3	1/3	3/5	1/5	1/5	1/2	1/3	1/2	1/2	1/3
61	144	1/3	2/5	2/5	2/5	1/2	1/5	0	1/3	1/3	2/5	2/3	3/5
62	144	1/5	1/3	2/5	2/5	1/2	1/5	1/5	1/3	1/3	1/2	1/2	1/2
63	144	1/5	1/5	1/3	2/5	1/2	0	1/5	1/2	1/3	1/3	1/2	1/3
64	144	1/5	1/5	1/5	1/3	1/2	0	0	1/2	1/5	1/3	2/5	1/5
65	144	1/3	1/3	2/5	2/5	3/5	0	1/5	3/5	2/5	1/3	2/3	1/5
66	144	1/5	1/5	1/5	1/3	1/3	0	0	1/3	1/5	1/3	2/5	2/5
67	200	1/5	2/5	2/5	1/2	1/2	1/5	1/3	1/3	1/3	2/3	1/2	1/2
68	200	1/5	1/5	2/5	1/2	1/2	0	1/3	1/2	2/5	1/3	1/2	2/5
69	205	1/5	1/3	1/3	1/3	2/5	1/5	0	1/3	1/5	1/3	1/2	1/2
70	220	1/3	1/3	1/3	2/5	2/5	0	0	2/5	1/3	2/5	2/3	1/2
71	220	1/5	1/5	1/3	1/3	1/3	0	1/5	1/3	1/3	1/5	1/2	1/3
72	225	1/3	1/3	1/3	1/3	2/5	1/5	1/5	1/5	2/5	2/5	1/2	1/2
73	225	1/5	1/3	1/3	1/3	2/3	1/5	1/5	3/5	1/3	1/2	1/2	1/3

Table 9 continued

♯	Length	θ_1	θ_2	θ_3	θ_4	θ_5	σ_{12}	σ_{23}	σ_{34}	σ_{45}	σ_{51}	σ_{13}	σ_{24}
74	240	2/5	2/5	2/5	1/2	1/2	0	1/3	1/2	2/5	1/5	3/5	1/3
75	240	1/5	1/5	1/5	1/2	1/2	0	1/5	1/2	1/5	1/3	1/3	1/3
76	240	1/3	1/3	2/5	2/5	1/2	1/5	1/5	1/3	1/3	1/3	1/2	2/5
77	240	1/5	1/3	1/3	2/5	1/2	1/5	0	2/5	1/5	2/5	1/2	1/2
78	240	1/5	1/5	1/3	1/3	1/2	0	1/5	1/2	1/3	1/3	1/2	1/5
79	252	1/3	1/3	1/3	1/3	2/3	0	1/5	2/3	1/3	2/5	3/5	1/5
80	300	1/3	1/3	1/3	1/3	3/5	0	0	3/5	1/3	1/3	2/3	1/5
81	300	1/5	1/3	1/3	1/3	1/3	1/5	0	1/5	1/5	1/3	1/2	3/5
82	360	1/3	2/5	2/5	1/2	1/2	1/5	0	1/3	1/3	1/2	2/3	2/3
83	360	1/5	1/3	2/5	1/2	1/2	1/5	1/5	1/3	1/3	1/2	1/2	1/2
84	360	1/5	1/5	1/3	1/2	1/2	0	1/5	1/2	1/3	2/5	1/2	2/5
85	400	1/3	1/3	1/3	2/5	1/2	0	0	1/2	1/3	2/5	2/3	2/5
86	400	1/5	1/3	1/3	1/3	1/2	1/5	0	1/2	1/5	1/3	1/2	1/3
87	432	1/3	1/3	1/3	1/3	1/3	0	1/5	1/3	1/3	1/5	3/5	1/3
88	480	2/5	2/5	1/2	1/2	1/2	0	1/5	1/2	1/2	1/3	4/5	1/2
89	480	1/5	1/5	1/2	1/2	1/2	0	1/3	1/2	1/2	2/5	2/3	1/2
90	576	1/3	1/3	1/3	1/3	1/2	0	1/5	1/2	1/3	1/5	3/5	1/5
91	580	1/5	2/5	1/2	1/2	1/2	1/5	1/5	2/5	1/3	3/5	2/3	1/2
92	600	1/3	1/3	2/5	1/2	1/2	0	1/5	1/2	2/5	1/3	2/3	2/5
93	600	1/5	1/3	1/3	1/2	1/2	1/5	0	1/3	1/5	1/2	1/2	2/3
94	900	1/3	2/5	1/2	1/2	1/2	1/5	1/5	1/3	2/5	2/5	3/5	1/2
95	900	1/5	1/3	1/2	1/2	1/2	1/5	1/5	1/3	2/5	1/2	2/3	3/5
96	936	1/3	1/3	1/3	1/2	1/2	0	1/5	1/2	1/3	1/3	3/5	1/3
97	1200	2/5	1/2	1/2	1/2	1/2	1/5	0	1/3	2/5	1/2	4/5	2/3
98	1200	1/5	1/2	1/2	1/2	1/2	1/3	0	1/3	1/5	1/2	2/3	2/3
99	1440	1/3	1/3	1/2	1/2	1/2	0	1/5	1/2	1/2	2/5	4/5	1/2
100	2160	1/3	1/2	1/2	1/2	1/2	1/5	0	2/5	1/3	1/2	4/5	3/5
101	3072	1/2	1/2	1/2	1/2	1/2	0	1/5	1/2	1/2	1/3	4/5	2/5

with different constraints on the parameters X, Y, Z, W and the off-diagonal elements V_x, V_y, V_z, V_w, V_5.

For orbits $1-8$ in the list the tuples of matrices were constructed explicitly, using notations $X=\exp(i\pi\, x)$, $Y=\exp(i\pi\, y)$, $Z=\exp(i\pi\, z)$.

Orbit 1:

$$M_1=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad M_2=\begin{pmatrix}X&0\\0&X^{-1}\end{pmatrix},\quad M_3=\begin{pmatrix}Y&0\\0&Y^{-1}\end{pmatrix},$$
$$M_4=\begin{pmatrix}Z&0\\0&Z^{-1}\end{pmatrix},\quad M_5=\begin{pmatrix}0&-(X\,Y\,Z)^{-1}\\X\,Y\,Z&0\end{pmatrix}. \tag{15}$$

Table 10 Ten exceptional orbits

♯	Length	θ_1	θ_2	θ_3	θ_4	θ_5	σ_{12}	σ_{23}	σ_{34}	σ_{45}	σ_{51}	σ_{13}	σ_{24}
0		x	y	z	w	$x+y+$ $z+w$	$x+y$	$y+z$	$z+w$	$x+y$ $+z$	$y+$ $z+w$	$x+z$	$y+w$
1	4	1/2	x	y	z	1/2	1/2	$x+y$	$y+z$	1/2	$x+y+z$	1/2	$x+z$
2		1/2	1/2	x	1/2	1/2	$x+y$	1/2	1/2	y	z	1/2	$x+z$
3	9	x	x	x	x	2/3	$2x$	$2x$	1/3	$3x$	1/2	$2x$	1/3
4	12	x	x	$2x+1$	x	x	1/3	1/2	1/2	1/3	$2x$	$3x+1$	$2x$
5	105	2/7	2/7	2/7	2/7	2/7	1/3	1/7	1/7	1/7	1/3	1/3	1/2
6	105	4/7	4/7	4/7	4/7	4/7	1/3	5/7	5/7	5/7	1/3	1/3	1/2
7	105	6/7	6/7	6/7	6/7	6/7	1/3	3/7	3/7	3/7	1/3	1/3	1/2
8	192	1	1	1	1	1	1/3	1/3	1/3	1/3	1/3	1/5	1/5
A	1	0	0	0	0	0	1	1	1	1	1	1	1

The length of this orbit is 4.

Orbit 2:

$$M_1=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad M_2=\begin{pmatrix}0&-X\,Y\\(X\,Y)^{-1}&0\end{pmatrix},\quad M_3=\begin{pmatrix}X&0\\0&X^{-1}\end{pmatrix},$$
$$M_4=\begin{pmatrix}0&Y Z^{-1}\\-Y^{-1}Z&0\end{pmatrix},\quad M_5=\begin{pmatrix}0&-Z^{-1}\\Z&0\end{pmatrix},\tag{16}$$

where $x,\ y,\ z$ must be rational numbers. The length of this orbit is

$$\frac{u^2\,v^2\prod_{p\geq 3}(1-p^{-2})}{1+\delta_{u,1}(1-\delta_{v,1})},$$

where u is the denominator of x, v is the smallest common denominator of values $(u\,y)$ and $(u\,z)$, and p is an odd prime divisor of v; the denominator of this equation is 2 for $u=1$ and $v\geq 2$, and 1 otherwise.

In orbits 1 and 2 all matrices belong to the dihedral group.

Orbit 3:

$$M_1=\begin{pmatrix}X&0\\1&X^{-1}\end{pmatrix},\quad M_2=\begin{pmatrix}X&0\\1&X^{-1}\end{pmatrix},\quad M_3=\begin{pmatrix}X&0\\1&X^{-1}\end{pmatrix},$$
$$M_4=\begin{pmatrix}X^{-1}&-1\\0&X\end{pmatrix},\quad M_5=\begin{pmatrix}-1-X^2&X^3\\-X-X^{-1}-X^{-3}&X^2\end{pmatrix}.\tag{17}$$

Orbit 4:

$$M_1=\begin{pmatrix}X&0\\-1&X^{-1}\end{pmatrix},\quad M_2=\begin{pmatrix}X^{-1}&1\\0&X\end{pmatrix},\quad M_3=\begin{pmatrix}-X^2&0\\X+X^{-1}&-X^{-2}\end{pmatrix},$$
$$M_4=\begin{pmatrix}X^{-1}&1\\0&X\end{pmatrix},\quad M_5=\begin{pmatrix}X&0\\-1&X^{-1}\end{pmatrix}.\tag{18}$$

Orbits 5, 6, 7 can be written in the same form:

$$M_1=\begin{pmatrix}s&0\\0&s^6\end{pmatrix},$$
$$M_2=\begin{pmatrix}s^6(s-1)^5(1+s)^3(1+s^2)^2/7&s^5(1+s^2)/7\\s(s-1)^4(1+s)^3&s^3(1-s)^5(1+s)^3(1+s^2)^2/7\end{pmatrix},$$
$$M_3=M_1^2\,M_2\,M_1^{-2},\quad M_4=M_1^{-3}\,M_2\,M_1^3,\quad M_5=M_1^{-1}\,M_2\,M_1,\tag{19}$$

where s is one of the three seventh roots of unity: $s=\exp(2\,i\pi/7)$, $s=\exp(4\,i\pi/7)$ or $s=\exp(6\,i\pi/7)$. Replacing $s\to s^{-1}$ yields the same matrices up to a common conjugation.

Orbit 8:

$$M_1 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} \frac{-1-\sqrt{5}}{2} & 1 \\ \frac{-3+\sqrt{5}}{2} & \frac{-3+\sqrt{5}}{2} \end{pmatrix},$$
$$M_4 = \begin{pmatrix} \frac{1-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} \\ \frac{-3+\sqrt{5}}{2} & \frac{-5+\sqrt{5}}{2} \end{pmatrix}, \quad M_5 = \begin{pmatrix} \frac{-1-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} \\ -1 & \frac{-3+\sqrt{5}}{2} \end{pmatrix}. \tag{20}$$

The orbit in $\mathcal{S}^F{}_5$, which is called A in the list, consists of one element, turns out to be inconsistent: all θ's are equal to 0 and all σ's are equal to 1. It is easy to try to reconstruct the matrix tuple from it and confirm that it is impossible. Therefore this signature must be excluded.

It is not surprising that only one signature turned out to be inconsistent, because the braid group actions are originally defined for tuples of matrices. Thus the result of braid group action on an inconsistent signature can only coincide with another signature from any list by accident only.

10 List of signatures of $\mathcal{M}_F{}^{(5)}$'s

Here we present the list of signatures of $\mathcal{M}_F{}^{(5)}$'s. From this list we omit the $\mathcal{M}_F{}^{(5)}$'s which are obtained from $\mathcal{M}_F{}^{(4)}$'s by addition of the unit matrix, and the triangular $\mathcal{M}_F{}^{(5)}$'s.

For each symmetry class of orbits under (3), (4) and (5) we present only one the representative orbit, and for each orbit—only one element (we call it start element, though every element of an orbit can be chosen as the start element).

For each signature, we present not all 20 cells, but only all θ's and seven of the σ's, because it is enough to reconstruct the tuple of matrices.

We divide the orbits from this list into the following types:

- A: Orbit 1. A dihedral orbit of length 4 with arbitrary parameters.
- B: Orbit 2. A dihedral orbit with rational parameters. Its length depends on parameters and can be arbitrarily large.
- C: Orbits 3 and 4 with one parameter.
- D: Orbits 9, 10, 11. Tetrahedral orbits.
- E: Orbits 12–30. Octahedral orbits.
- F: Orbits 31–101. Icosahedral orbits.
- G: Orbits 5, 6, 7.
- H: Orbit 8.
- N: The orbits obtained from $\mathcal{M}_F{}^{(4)}$ by addition of the unit matrix. These orbits are omitted in this list.

Orbit 1 is described in [5].

Orbits 8, 11, 17, 19, 20, 23, 24, 25, 26, 27, 28, 29, 30, 35, 36, 37, 40, 46, 47, 52, 53, 55, 56, 58, 59, 65, 66, 67, 68, 69, 70, 71, 74, 75, 76, 78, 80, 81, 82, 83, 84, 88, 89, 91, 92, 93, 94, 95, 97, 98, 100, 101 (52 orbits in total) appear in [4].

Also, we found that orbits 42 and 52 from Table 2 in [4] have lengths 432 and 1440 respectively. We think that in [4] possibly there are misprints in lengths of these two orbits because these lengths repeat lengths in the adjacent rows in the table. If one fix it, these orbit coincide with orbits 87 and 99 from the Table 9 in the present paper, respectively.

11 Conjecture about classification of $\mathcal{M}^{(n)}$ for any n

We conject that for any $n \geq 3$ there exist only the following finite orbits (including cases $n = 4, 5$ which are already classified):

Type I. Orbits of *triangular* $\mathcal{M}^{(n)}$'s, where all matrices have a common eigenvector (not considered in this paper; need a separate classification),

Types A, B, D, E, F. Orbits of these types exist for all n. All matrices in these orbits belong to a subgroup of $SU(2)$, including:

- A: belonging to the dihedral group, possibly infinite. Each element of such orbit contains two matrices with zeros on the main diagonal and $n-2$ diagonal matrices; there are $n-2$ arbitrary parameters. Length of the orbit is 2^{n-3}.
- B: belonging to any finite dihedral group. There are $\lfloor n/2 \rfloor - 1$ sub-types of such orbits, we denote them with integer number $m \in [2, \lfloor n/2 \rfloor]$. Each element of an orbit of each sub-type contains $2m$ matrices with zeros on the main diagonal, and $n - 2m$ diagonal matrices; there are $n-2$ rational parameters. The length of the orbit is by order of magnitude as large as the common denominator of all parameters raised to the power $2m-2$.
- D: tetrahedral orbits: all matrices belong to the tetrahedral group.
- E: octahedral orbits: all matrices belong to the octahedral group.
- F: icosahedral orbits: all matrices belong to the icosahedral group.

Type C: this type exists only for $n = 4, 5, 6$ and contains one parameter. For $n = 6$ it has the form

$$
\begin{aligned}
M_1 &= \begin{pmatrix} X & 0 \\ 1 & X^{-1} \end{pmatrix}, \quad M_2 = \begin{pmatrix} X^{-1} & -1 \\ 0 & X \end{pmatrix}, \quad M_3 = \begin{pmatrix} -X^{-1} & 1 \\ 0 & -X \end{pmatrix}, \\
M_4 &= \begin{pmatrix} X & 0 \\ 1 & X^{-1} \end{pmatrix}, \quad M_5 = \begin{pmatrix} X^{-1} & -1 \\ 0 & X \end{pmatrix}, \quad M_6 = \begin{pmatrix} X^{-1} & -1 \\ 0 & X \end{pmatrix},
\end{aligned} \tag{21}
$$

and for $n = 5, 4$ reductions of this orbit.

Types G and H exist for $n = 4, 5$. For $n = 5$ there are exceptional orbits (19) and (20), and for $n = 4$ their reductions.

Type K: for $n = 3$ arbitrary tuple of matrices.

Type M: For $n = 4$ in every orbit of types A, B, C, D, E, F, G, H we can replace $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ with

$$
\{\theta_1 - \delta, \theta_2 - \delta, \theta_3 - \delta, \theta_4 - \delta\}, \quad \delta = \frac{\theta_1 + \theta_2 + \theta_3 + \theta_4}{2},
$$

or with

$$\{\theta_1 - \delta,\ \theta_2 - \delta,\ \theta_3 - \delta,\ \delta - \theta_4\},\quad \delta = \frac{\theta_1 + \theta_2 + \theta_3 - \theta_4}{2}.$$

This type is completely considered in [3], and such replacements of θ's match the Okamoto transformations.

Type N: for any orbit we can trivially increase n by addition of the unit matrix, or by addition of the minus unit matrix and multiplying any other matrix by -1.

The reason why we believe in this conjecture is that for every higher n the diversity of finite orbits becomes lower. This is because more and more conditions force orbits to be finite. Moreover, each finite orbit for higher n ($n \geq 6$) must have a reduction—a finite orbit for lower n (e.g. $n = 5$). We checked this conjecture with a non-exact computer search, but without conclusive results.

12 Discussion

By this algorithm we can construct the list of $\mathcal{M}_F{}^{(n)}$'s from a the list of $\mathcal{M}_F{}^{(n-1)}$'s. Thus, starting from the list of $\mathcal{M}_F{}^{(4)}$'s, we can step by step get lists for all n.

We classified the finite orbits of monodromies under braid group actions for five branching points in the Fuchsian system for 2×2 matrices, and made a conjecture for such classification for six and more branching points.

Some of the orbits of five monodromy matrices are listed in Calligaris and Mazzocco paper [4], and also in Diarra's [6], Girand's [5] and Tsuda's [7], but the full list turned out to be rather bigger. E.g. new type orbits (5, 6, 7 in Table 9) were found.

The method of [4] is similar to the method presented in this paper. The variables $p_{a,b\ldots}$ in [4] correspond to $2\cos\left(\pi\,\sigma_{a,b\ldots}\right)$ in the present paper.

However, many orbits from Table 9 in the present paper are absent in [4]. One reason may be that in [4] it is declared that the authors used only the exceptional orbits from [3] for construction. However, the orbit number 9 from the present paper, which is considered in detail in Sects. 5, 6, construction of which needs not only the exceptional orbits from [3], but also orbits with parameters, is absent in [4]. Another reason may be a limitation of the arithmetical method employed in [4], which can handle only explicit numbers in radical form, e.g. $\sqrt{2}$ and $\left(1 + \sqrt{5}\right)/2$. Therefore, the orbits 5, 6, 7 from the present paper, for which p values include seventh roots of unity, which cannot be expressed in radicals, are also absent in [4].

The *triangular* cases where all monodromy matrices have a common eigenvector and can be simultaneously put into lower triangular form, were classified by Cousin and Moussard in [8]. We left these results outside of the present paper, because the method used in this paper is not applicable to such cases.

Although the proof uses the computer, it is an exact proof, because the computer was tasked with searching through a finite number of options using integer arithmetic (although too large for a human to check in a reasonable amount of time). The program was monitored for abnormal situations and no such situation ever happened. In the end the procedure finished regularly after exhausting of the possibility space. The proof of the conjecture for six and more matrices must be analytical, but we expect it to be not too complicated, because under our conjecture the list of orbits for six and more matrices is rather uniform.

Each algebraic solution of the Garnier system corresponds a finite orbit of $\mathcal{M}^{(n)}$. That's why we think that each orbit of $\mathcal{M}_F{}^{(n)}$ with its tuple of exact values of θ's may generate one algebraic solution of the Garnier system, or a finite number of algebraic solutions corresponding to the symmetry group of this $\mathcal{M}_F{}^{(n)}$. The solutions for which one of θ's differs by 2 from a given solution, can be obtained from it with an algebraic transformation, analogous to Bäcklund transformations group, see [9].

Conclusions

The finite monodromies of the Fuchsian system for five 2×2 matrices have been classified, except for the case in which all monodromy matrices have a common eigenvector.

The conjecture about such classification for six and more matrices is formulated.

Acknowledgements The research described in this paper was supported by the National Academy of Sciences of Ukraine (Project No. 0117U000238). The author would like to thank Dr. O. Lisovyy for formulation of the problem, and Dr. N. Iorgov for fruitful discussions.

Data Availability Statement The authors confirm that the data supporting the findings of this study are available within the article.

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